



Malcolm, A. (2018). The involution width of finite simple groups.
Journal of Algebra, 493, 297–340.
<https://doi.org/10.1016/j.jalgebra.2017.08.036>

Peer reviewed version

Link to published version (if available):
[10.1016/j.jalgebra.2017.08.036](https://doi.org/10.1016/j.jalgebra.2017.08.036)

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THE INVOLUTION WIDTH OF FINITE SIMPLE GROUPS

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ABSTRACT. For a finite group generated by involutions, the involution width is defined to be the minimal $k \in \mathbb{N}$ such that any group element can be written as a product of at most k involutions. We show that the involution width of every non-abelian finite simple group is at most 4. This result is sharp, as there are families with involution width precisely 4.

1. INTRODUCTION

Let G be a finite group generated by involutions. The involution width, denoted $\text{iw}(G)$, is defined to be the minimal $k \in \mathbb{N}$ such that any element of G can be written as a product of at most k involutions. It is well known that all non-abelian finite simple groups are generated by their involutions. Furthermore, it was proved by Liebeck and Shalev ([39], 1.4) that there exists an absolute constant N that bounds the involution width of all finite simple groups. The purpose of this paper is to obtain the minimal value for N .

The involution width was first considered in the case of special linear groups by Wonenburger [57], and later by Gustafson et. al. [27] and Knüppel and Nielsen [31]. Further work has been completed on orthogonal groups (Knüppel and Thomsen, [32]) and the exceptional group $F_4(K)$ (Austin, [2]).

More recently, various problems have been resolved involving the related notions of real or strongly real groups. An element $x \in G$ is real if $x^g = x^{-1}$ for some $g \in G$. Furthermore x is strongly real if g can be taken to be an involution. We call the group G (strongly) real if all of its elements are (strongly) real. It follows easily that $x \in G$ is strongly real if and only if x is a product of 2 involutions and so the strongly real groups are precisely those of involution width at most 2. The classification of strongly real, finite simple groups was completed in 2010 after work by a number of authors (see [22, 30, 46, 51, 18] and Theorem 2.1 below). The work of this paper addresses the remaining finite simple groups and is summarised by the following theorem.

Theorem 1. *Every non-abelian finite simple group has involution width at most 4.*

Note that the upper bound 4 is sharp, as certain families (for example $PSL_n(q)$ such that $n, q \geq 6$ and $\gcd(n, q-1) = 1$) do have involution width 4 (see Theorem 3.6).

The involution width is one of a number of width questions that have been considered about simple groups in recent literature. For example, [34] settles the longstanding conjecture of Ore that the commutator width of any finite non-abelian simple group G is exactly 1. Also, G is generated by its set of squares and the width in this case is 2 [35]. More generally, given any two non-trivial words w_1, w_2 , if G is of large enough order then $w_1(G)w_2(G) \supseteq G \setminus \{1\}$ [26].

Here are some remarks on the proof of Theorem 1. For alternating groups we find the involution width directly by studying the disjoint cycle decomposition of elements. For groups G of Lie type we adopt a different approach: we aim to find particular regular semisimple elements $x, y \in G$ such that x and y are strongly real and $G \setminus \{1\} \subseteq x^G y^G$. It

Date: November 9, 2017.

2010 Mathematics Subject Classification. Primary 20D6; Secondary 20C33 .

then follows that every element is a product of at most 4 involutions. To do this we make extensive use of the character theory of finite groups of Lie type, building on methods first seen in [41] and [25]. Substantial difficulties are faced in the case of unitary groups, where we develop the theory of minimal degree characters using dual pairs (Sec. 4). Similarly problematic are a number of exceptional groups of Lie type and in these instances we use an inductive approach, restricting to subgroups of G for which the involution width is known (Sec. 5).

Naturally the involution width problem can be generalised to elements of order p , for any fixed prime p . This has been resolved in the case of alternating groups and work on the simple groups of Lie type will be forthcoming.

Acknowledgements: This paper forms part of work towards a PhD degree under the supervision of Martin Liebeck at Imperial College London, and the author wishes to thank him for his direction and support throughout. The author is also very thankful to Pham Huu Tiep for helpful advice on the character theory of finite simple groups, and to Gunter Malle for suggesting improvements to an earlier version of the paper. This work was supported by an EPSRC studentship.

2. ALTERNATING AND SPORADIC GROUPS

2.1. Alternating Groups. We first consider the simple alternating groups A_m , $m \geq 5$. Conjugacy in A_m is easily understood: classes in the full symmetric group S_m are indexed by cycle type. These types correspond to partitions of m and an S_m -class splits in A_m if and only if the partition consists of distinct odd integers. We can find the involution width directly by studying the permutations and, unlike later work on groups of Lie type, require no representation theory.

For reference, we first note the classification of strongly real finite simple groups.

Theorem 2.1. ([22, 30, 46, 51, 18]). *A finite non-abelian simple group is strongly real if and only if it is one of the following*

- (1) $PSp_{2n}(q)$ where $q \not\equiv 3 \pmod{4}$ and $n \geq 1$;
- (2) $P\Omega_{2n+1}(q)$ where $q \equiv 1 \pmod{4}$ and $n \geq 3$;
- (3) $P\Omega_9(q)$ where $q \equiv 3 \pmod{4}$;
- (4) $P\Omega_{4n}^+(q)$ where $q \not\equiv 3 \pmod{4}$ and $n \geq 3$;
- (5) $P\Omega_{4n}^-(q)$ where $n \geq 2$;
- (6) $P\Omega_8^+(q)$ or ${}^3D_4(q)$;
- (7) $A_5, A_6, A_{10}, A_{14}, J_1, J_2$.

Evidently, all but four of the alternating groups are not strongly real. We find that products of three involutions are needed in general.

Definition 2.2. Let G be a finite group generated by involutions and let $g \in G$. The involution width of g , denoted $\text{iw}(g)$, is the minimal $k \in \mathbb{N}$ such that g can be written as a product of k involutions.

Theorem 2.3. *The involution width of A_m ($m \geq 5$) is at most 3.*

Proof. Let $g \in A_m$ and write g as a product of disjoint cycles $g = c_1 \dots c_k$ where $c_i \in S_n$. Recall that $n_0 + n_2 \equiv 0 \pmod{2}$, where n_j denotes the number of cycles of g of length congruent to $j \pmod{4}$. Denote the length of a cycle by $|c_i|$. We consider single cycles and also pairs from this decomposition and show how they can be written as a product of at most 3 involutions.

First consider $c \in \{c_1, \dots, c_k\}$ such that $|c| = n \equiv 1 \pmod{4}$. Without loss of generality we assume that $c = (1 \dots n)$. Define

$$\begin{aligned} x_1 &:= (1 \ n)(2 \ n-1) \dots \left(\frac{n-1}{2} \ \frac{n+3}{2}\right), \\ x_2 &:= (2 \ n)(3 \ n-1) \dots \left(\frac{n+1}{2} \ \frac{n+3}{2}\right). \end{aligned}$$

Then x_1 and x_2 are involutions in A_m and $c = x_1 \cdot x_2$.

Next consider a pair of disjoint cycles $c, c' \in \{c_1, \dots, c_k\}$ of even lengths n and n' respectively. Without loss of generality we assume these have the form $c = (1 \dots n)$ and $c' = (1' \dots n')$. For $n > 2$ define

$$\begin{aligned} y_1 &:= (1 \ n-1)(2 \ n-2) \dots \left(\frac{n-2}{2} \ \frac{n+2}{2}\right), \\ y_2 &:= (1 \ n)(2 \ n-1) \dots \left(\frac{n}{2} \ \frac{n+2}{2}\right), \\ z_1 &:= (1 \ n)(2 \ n-1) \dots \left(\frac{n}{2} \ \frac{n+2}{2}\right), \\ z_2 &:= (2 \ n)(3 \ n-1) \dots \left(\frac{n}{2} \ \frac{n+3}{2}\right). \end{aligned}$$

When $n = 2$ define $y_1 = z_2 = 1$ and $y_2 = z_1 = (1 \ 2)$. Furthermore, define y'_1, y'_2 etc to be the permutations as above, but acting on the letters $1', \dots, n'$. Evidently all of the above elements have order dividing two, but note that they are not necessarily even permutations. In fact, when $n \equiv 0 \pmod{4}$, $y_2, z_1 \in A_m$ and $y_1, z_2 \in S_m \setminus A_m$. If $n \equiv 2 \pmod{4}$ then the reverse situation holds. Naturally the same can be said for y'_1 etc.

First assume that $n \equiv n' \equiv 0 \pmod{4}$. Then $c \cdot c' = (y_1 y_2) \cdot (y'_1 y'_2)$. But y_2 and y'_1 contain no common letters and hence commute. We therefore rewrite $cc' = (y_1 y'_1) \cdot (y_2 y'_2)$, a product of two involutions, $y_1 y'_1$ and $y_2 y'_2$ in A_m . In an identical manner we write $cc' = (z_1 z'_1) \cdot (z_2 z'_2)$ when $n \equiv n' \equiv 2 \pmod{4}$, and $cc' = (y_1 z'_1) \cdot (y_2 z'_2)$ when $n \equiv 0 \pmod{4}$ and $n' \equiv 2 \pmod{4}$. Each bracketed expression is an even permutation and hence in all cases we have written cc' as the product of two involutions in A_m .

Next we consider cycles of length $3 \pmod{4}$. In particular, let $c, c' \in \{c_1, \dots, c_k\}$ be disjoint cycles of lengths n and n' respectively such that $n \equiv n' \equiv 3 \pmod{4}$. Assuming that $c = (1 \dots n)$ and $c' = (1' \dots n')$, it follows that $c = x_1 x_2$ and $c' = x'_1 x'_2$, where x_1, x_2 etc are as above. Note however, that unlike the case above of $n \equiv 1 \pmod{4}$, here the x_i are in $S_n \setminus A_n$. But x_2 and x'_1 commute and it follows that cc' is indeed the product of the two involutions $x_1 x'_1$ and $x_2 x'_2$ in A_m .

Lastly consider the cycle $c = (1 \dots n)$ such that $n \equiv 3 \pmod{4}$, alone. If we assume additionally that $n \geq 7$, then we can easily write c as the product of three involutions in A_m . Namely

$$c = \left(x_1 \left(\frac{n-1}{2} \ \frac{n+3}{2}\right)\right) \cdot \left(\left(\frac{n-1}{2} \ \frac{n+3}{2}\right)(2 \ n)\right) \cdot ((2 \ n)x_2). \quad (1)$$

Note that this analysis has ignored single cycles of length 3. This, as we shall see below, is because such cycles must be considered within the context of the whole element g .

Let us now return to our original element $g = c_1 \dots c_k$. As the c_i commute, we assume that cycles of lengths congruent to $3 \pmod{4}$ appear at the end of the decomposition, and are ordered by increasing length. In particular, $|c_k| = 3$ if and only if all cycles of length $3 \pmod{4}$ are in fact of length 3 (and at least one of these exists).

Assume for now that n_3 , the number of cycles of length congruent to $3 \pmod{4}$, is even. Then by the above, each cycle or pair of cycles in g can be decomposed as products of two involutions in A_m . These decompositions have the form $c_i = t_{1_i} t_{2_i}$ or $c_i c_{i+1} = t_{1_i} t_{2_i}$,

where $t_{i_j} \in A_m$ denote the involutions. These involutions then commute where necessary to yield

$$g = (t_{1_1} t_{2_1}) \dots (t_{1_p} t_{2_p}) = (t_{1_1} \dots t_{1_p}) \cdot (t_{2_1} \dots t_{2_p}),$$

where $p = n_1 + (n_0 + n_2)/2 + n_3/2$. Evidently $(t_{1_1} \dots t_{1_p})$ and $(t_{2_1} \dots t_{2_p})$ are involutions in A_m and hence g is strongly real.

Now consider the remaining case, where n_3 is odd. Firstly note that gc_k^{-1} is strongly real by the above method. In particular there exist involutions $t_{i_j} \in A_m$ such that

$$gc_k^{-1} = (t_{1_1} \dots t_{1_q}) \cdot (t_{2_1} \dots t_{2_q}),$$

where $q = n_1 + (n_0 + n_2)/2 + (n_3 - 1)/2$. If $|c_k| > 3$, then c_k can be written as a product of three involutions as shown by (1). Denote this decomposition $c_k = s_1 s_2 s_3$, for involutions $s_i \in A_m$. It then follows that $g = (t_{1_1} \dots t_{1_q} s_1) \cdot (t_{2_1} \dots t_{2_q} s_2) \cdot s_3$ and $\text{iw}(g) \leq 3$.

This leaves the case where $|c_k| = 3$. Write $c_k = (1\ 2\ 3) = (1\ 2)(1\ 3)$. If $g = c_k$ then $c_k = ((1\ 2)(4\ 5)) \cdot ((4\ 5)(1\ 3))$ and g is strongly real. Suppose instead that $gc_k^{-1} \neq 1$. Then one of the elements $(t_{1_1} \dots t_{1_q})$ and $(t_{2_1} \dots t_{2_q})$ is non-trivial, say the latter. Let $(i\ j)$ be a transposition in its cycle decomposition. This commutes with c_k and hence we can reorder the transpositions to give

$$g = ((t_{1_1} \dots t_{1_q}) \cdot (t_{2_1} \dots t_{2_q} (i\ j)(1\ 2))) \cdot ((i\ j)(1\ 3)).$$

This is a product of three involutions in A_m . □

It is worth noting the following corollary of the proof above.

Corollary 2.4. *Let $g \in A_m$ for $m \geq 5$. Suppose that g has at least 2 fixed points or an even number of cycles with length 3 mod 4. Then g can be written as the product of 2 involutions in A_m .*

Proof. From the proof of Theorem 2.3 it is sufficient to show that if c is a cycle of length $n \equiv 3 \pmod{4}$ such that $c \in A_{n+k}$ for $k \geq 2$, then $c = t_1 t_2$ for involutions $t_i \in A_{n+k}$.

Without loss of generality, take $c = (1\ 2 \dots n)$ with fixed points $n+1$ and $n+2$. We can write c as a product of two involutions as follows

$$\begin{aligned} (1\ 2 \dots n) &= \left((1\ n)(2\ n-1) \dots \left(\frac{n-1}{2} \frac{n+3}{2} \right) (n+1\ n+2) \right) \\ &\quad \cdot \left((n+1\ n+2)(2\ n)(3\ n-1) \dots \left(\frac{n+1}{2} \frac{n+3}{2} \right) \right). \end{aligned}$$

□

2.2. Sporadic Groups. In this section we prove the following result.

Theorem 2.5. *Let G be a sporadic finite simple group. If $G \in \{J_1, J_2\}$ then G has involution width 2, otherwise G has width 3.*

It was found by Suleiman ([51], see also Theorem 2.1) that only two sporadic groups, namely J_1 and J_2 are strongly real. Furthermore, [51] lists the non-strongly real classes of every other sporadic group. We compute the involution width of these classes by calculating the structure constants, which are defined as follows.

Definition 2.6. Let G be a finite group and let C_i , $1 \leq i \leq m$ be conjugacy classes in G . For $g \in G$, define the *structure constant* $\eta(C_1, \dots, C_m, g^G)$ to be the number of m -tuples $(g_1, \dots, g_m) \in C_1 \times \dots \times C_m$ such that $g_1 \dots g_m = g$.

The structure constants can be computed using the following well known formula.

Theorem 2.7. ([1], 1.10). *Let G be a finite group and let C_i , $1 \leq i \leq m$ be conjugacy classes in G . For $g \in G$,*

$$\eta(C_1, \dots, C_m, g^G) = \frac{|G|}{|C_G(g_1)| \dots |C_G(g_m)|} \sum_{\chi} \frac{\chi(g_1) \dots \chi(g_m) \chi(g^{-1})}{\chi(1)^{m-1}},$$

where the sum is over all the irreducible characters χ of G .

Naturally $\eta(C_1, \dots, C_m, g^G)$ is a non-negative integer. However for our purposes we will only be concerned with showing that $\eta(C_1, \dots, C_m, g^G) \neq 0$. Therefore to simplify proceedings, we will predominantly compute

$$\kappa(C_1, \dots, C_m, g^G) = \sum_{\chi} \frac{\chi(g_1) \dots \chi(g_m) \chi(g^{-1})}{\chi(1)^{m-1}},$$

which we call the *normalised structure constant*.

Proof of Theorem 2.5:

The full character tables are known for all sporadic groups ([9]) and hence the structure constants can be calculated explicitly. We use the software package GAP [19] and a short function that checks whether $\eta(C_1, C_1, C_1, C_2)$ is nonzero for C_1 , a class of involutions and C_2 , a chosen non-strongly real class. For almost all the sporadic groups, we find that every such conjugacy class C_2 is contained in the cube of the $2A$ class. For the exceptions He , Co_2 , Fi_{22} , Fi_{23} and BM , we find instead that each class $C_2 \subset (2B)^3$. \square

3. NON-UNITARY CLASSICAL GROUPS

3.1. Preliminary Material. The remainder of this paper will be devoted to proving Theorem 1 for the simple groups of Lie type. Throughout this section, \mathbf{G} denotes a simple linear algebraic group over an algebraically closed field of characteristic $p > 0$. Let $F : \mathbf{G} \rightarrow \mathbf{G}$ be a Frobenius endomorphism. Then the set of fixed points $G := \mathbf{G}^F$ is the associated finite group of Lie type. Usually we take \mathbf{G} to be simply connected, in which case, with a small number of exceptions, G is quasisimple and $\overline{G} = G/Z(G)$ is the simple group of interest.

The general procedure for proving Theorem 1 for a simple group of Lie type \overline{G} is as follows. We aim to pick regular semisimple elements x and y in particular classes of G , such that

- (1) The projections $\overline{x}, \overline{y}$ are strongly real in the simple group \overline{G} .
- (2) $G \setminus Z(G) \subset x^G \cdot y^G$.

From (2) it will follow that $\overline{G} \setminus \{1\} \subset \overline{x}^{\overline{G}} \cdot \overline{y}^{\overline{G}}$ and hence $\text{iw}(\overline{G}) \leq \text{iw}(\overline{x}) + \text{iw}(\overline{y}) \leq 4$ by (1). The proof that $G \setminus Z(G) \subseteq x^G \cdot y^G$ involves calculating structure constants using Theorem 2.7. However unlike the sporadic case, complete character tables are currently unavailable for most groups of Lie type. We therefore rely substantially on estimates of character values. In particular, technical details of the characters of unitary groups form a substantial part of this work (see Sec. 4).

Before proceeding, we briefly recall some results on ordinary representations of finite groups of Lie type (see Carter [8] and Digne and Michel [13]).

Let \mathbf{T} be an F -stable maximal torus in \mathbf{G} and $\theta \in \text{Irr}(\mathbf{T}^F)$. One can define a corresponding Deligne-Lusztig character $R_{\mathbf{T}}^G(\theta)$ of the fixed point group $G = \mathbf{G}^F$ such that $R_{\mathbf{T}}^G(\theta) \in \mathbb{Z}[\text{Irr}(G)]$ ([8], 7.2). Moreover, for $\chi \in \text{Irr}(G)$, there is a pair (\mathbf{T}, θ) such that χ occurs in the decomposition of $R_{\mathbf{T}}^G(\theta)$.

We shall work in the setting of Lusztig series of characters. For this let \mathbf{G}^* be the reductive group in duality with \mathbf{G} , with corresponding Frobenius endomorphism $F^* : \mathbf{G}^* \rightarrow \mathbf{G}^*$.

The dual finite group of Lie type is then the fixed point set $G^* := \mathbf{G}^{*F^*}$ (see ([13], 13.10)). There is a bijective correspondence between G -conjugacy classes of pairs (T, θ) as defined above, and G^* -classes of pairs (T^*, s) , where $T^* \leq G^*$ is a maximal torus dual to T and $s \in T^*$ is semisimple ([13], 13.12). If (T^*, s) corresponds to (T, θ) in this manner, we re-label $R_T^G(\theta)$ as $R_{T^*}^{G^*}(s)$. The Lusztig series $\mathcal{E}(G, (s))$ is then defined to be the set of irreducible constituents of the $R_{T^*}^{G^*}(s)$ for T^* running over maximal tori of G^* containing s . We note the following lemma.

Lemma 3.1. ([25], 3.2). *Let $g \in G$ be semisimple and $\chi \in \text{Irr}(G)$ with $\chi(g) \neq 0$. Then there exists a maximal torus $T \ni g$ of G and $s \in T^* \leq G^*$ such that $\chi \in \mathcal{E}(G^F, (s))$.*

Of the characters of finite groups of Lie type, the best understood are the so-called unipotent characters. These are the elements of the Lusztig series $\mathcal{E}(G, (1))$.

Proposition 3.2. (Jordan decomposition of characters, ([13], 13.23)). *For any semisimple element $s \in G^*$, there is a bijection ψ_s from $\mathcal{E}(G, (s))$ to $\mathcal{E}(C_{G^*}(s)^{F^*}, (1))$. The degree of any character $\chi \in \mathcal{E}(G, (s))$ is given by the formula*

$$\chi(1) = |G^* : C_{G^*}(s)^{F^*}|_{p'} \cdot (\psi_s(\chi))(1).$$

Note that some caution is required when the centre of \mathbf{G} is disconnected. If the centre of \mathbf{G} is connected, then the group $C_{G^*}(s)$ is connected reductive for any semisimple $s \in G^*$. If however the centre of \mathbf{G} is disconnected then the definition of the unipotent characters $\mathcal{E}(C_{G^*}(s)^{F^*}, 1)$ needs to be generalised to disconnected groups ([13], 13.23).

Recall that for a prime p and a character $\chi \in \text{Irr}(G)$, we say that χ has p -defect zero if p does not divide $\frac{|G|}{\chi(1)}$. The study of p -defect zero characters plays an important role in this work and we will often use the following result of Brauer.

Theorem 3.3. ([29], 8.17). *Let p be a prime and $\chi \in \text{Irr}(G)$ a character of p -defect zero. Then $\chi(g) = 0$ whenever $p \mid o(g)$.*

We will predominantly apply the above theorem with respect to particular prime factors of $|G|$ known as primitive prime divisors.

Definition 3.4. Let $q = p^f$ for some prime p and let $n \geq 2$. Then a primitive prime divisor, denoted $\text{ppd}(q, n)$, is a prime r such that $r \mid (q^n - 1)$ but $r \nmid (q^k - 1)$ for $1 \leq k \leq n - 1$. A primitive prime divisor always exists for $n > 2$ by Zsigmondy's Theorem ([3], Thm. V) except when $(n, q) = (6, 2)$.

A final result that will be useful in proving step (2) of the procedure detailed above is the following, due to Gow.

Theorem 3.5. ([23], 2). *Let G be a finite simple group of Lie type and let L_1 and L_2 be conjugacy classes of G consisting of regular semisimple elements. Then any non-identity semisimple element of G is expressible as a product xy , where $x \in L_1$ and $y \in L_2$.*

First we consider the non-unitary classical groups, that is the linear, orthogonal and symplectic finite simple groups. These are denoted $PSL_n(q)$, $P\Omega_n^\pm(q)$ for n even, $P\Omega_n(q)$ for n odd and $PSp_n(q)$. We see that for some families, the involution width problem has already been considered and we quote results where appropriate.

3.2. Linear Groups. In [27] Gustavson et al. proved that the group of matrices in $GL_n(K)$ (K a field) of determinant ± 1 has involution width at most 4. The problem of the special linear group was then addressed by Knüppel and Nielsen [31].

Theorem 3.6. ([31], Thm. A). *Let $G = SL_n(q)$, $n \geq 3$. Then G has involution width 3 if and only if at least one of the following holds*

- (1) $n = 4$,
- (2) $q \in \{2, 3, 5\}$ and $n \not\equiv 2 \pmod{4}$,
- (3) $n = 3$, and either $\text{char}(\mathbb{F}_q) = 3$ or $x^2 + x + 1$ is irreducible over \mathbb{F}_q .

Otherwise G has involution width 4.

Corollary 3.7. *Let $\bar{G} = PSL_n(q)$, $n \geq 2$ (not $PSL_2(2)$ or $PSL_2(3)$).*

- (1) *Then $\text{iw}(\bar{G}) = 2$ if and only if $n = 2$ and $q \not\equiv 3 \pmod{4}$.*
- (2) *Suppose (n, q) satisfies at least one of the conditions 1-3 of Theorem 3.6 or $(n, q) = (2, 3 \pmod{4})$. Then $\text{iw}(\bar{G}) = 3$.*
- (3) *In any case, $\text{iw}(\bar{G}) \leq 4$.*

Proof. Part 1 follows from Theorem 2.1. Part 2 follows from Theorem 3.6 together with ([1], Thm 4.2), which shows that $C^3 = PSL_2(q)$ for any non-identity class C . Part 3 is immediate from Theorem 3.6. \square

3.3. Orthogonal Groups. The orthogonal groups are another classical family that have been considered in the literature. Building on a number of results we prove

Theorem 3.8. *Let $n \geq 7$ and let $\bar{G} = P\Omega_n^\pm(q)$ (n even) or $P\Omega_n(q)$ (n odd).*

- (1) *If q is odd then $\text{iw}(\bar{G}) \leq 3$.*
- (2) *If q is even then $\text{iw}(\bar{G}) \leq 4$.*

Proof. Firstly assume that q is odd. By ([32], 8.5), $\text{iw}(\Omega_n(q)) \leq 3$. The result then holds in the simple group after taking the natural map. Next assume that q is even (and hence $G = P\Omega_{2m}^\epsilon(q)$). By ([14], Thm. 3) we know that any $g \in G$ can be written as a product of two unipotent elements of G . But then, by ([46], Thm. 1.2), such unipotent elements are strongly real and hence it follows that $\text{iw}(g) \leq 4$. \square

3.4. Symplectic Groups. Recall from Theorem 2.1 that the symplectic groups $PSp_{2n}(q)$ are strongly real if $q \not\equiv 3 \pmod{4}$. The next result considers $PSp_{2n}(q)$ for all values of q . Note also that $PSp_2(q) \cong PSL_2(q)$ and thus by Corollary 3.7 we can assume that $n \geq 2$.

Theorem 3.9. *Let $G = Sp_{2n}(q)$ with $n \geq 2$. Then $\bar{G} = PSp_{2n}(q)$ has involution width at most 4.*

Proof. The work of Malle, Saxl and Weigel ([41], 2.3) shows that as long as $(n, q) \notin \{(2, 2), (2, 3), (3, 2)\}$ we can find regular semisimple elements $x, y \in G$ such that $x^G \cdot y^G \supseteq G \setminus Z(G)$. Such elements are contained in tori T_1 of order $q^n + 1$ and T_2 of order $q^n - 1$ respectively. The cases where $(n, q) \in \{(2, 2), (2, 3), (3, 2)\}$ are easily dealt with using the character tables in [9], so we assume (n, q) is not one of these from now on.

Let \bar{g} be the image of $g \in G$ under the canonical map $G \rightarrow \bar{G}$. By ([41], 2.3(c)), \bar{x} is strongly real so $\text{iw}(\bar{x}) = 2$. We claim that \bar{y} is also strongly real, from which the theorem will follow.

Now T_2 is a cyclic torus, conjugate to a Singer cycle $GL_1(q^n) < GL_n(q) < G$. Let $z \in T_2$ be a generator of the torus. We shall show that a conjugate of z is strongly real and thus so is z itself.

Let $e_1, \dots, e_n, f_1, \dots, f_n$ be a standard basis for the symplectic space, where $(e_i, f_j) = \delta_{ij}$ and $V_e = \langle e_1, \dots, e_n \rangle$, $V_f = \langle f_1, \dots, f_n \rangle$ are totally isotropic subspaces. Let $H \cong GL_n(q)$ be the stabiliser in G of V_e, V_f . Explicitly,

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \mid A \in GL_n(q) \right\}$$

and we let $z \in H$ have matrix representation $\begin{pmatrix} B & 0 \\ 0 & B^{-T} \end{pmatrix}$ for some $B \in GL_n(q)$.

Now let $t \in Sp_{2n}(q)$ be the map that sends $e_i \rightarrow f_i$ and $f_i \rightarrow -e_i$, $1 \leq i \leq n$. Evidently this map normalises the subgroup H and swaps V_e and V_f . Explicitly, t is given by the matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and thus conjugation of $z \in H$ is given by

$$t^{-1}zt = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-T} \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} B^{-T} & 0 \\ 0 & B \end{pmatrix}.$$

Clearly, \bar{t} is an involution and hence to show \bar{z} is strongly real it suffices to show that B is conjugate in $GL_n(q)$ to a symmetric matrix.

As B is a Singer cycle in $GL_n(q)$ it has a minimal polynomial f of degree n . Furthermore B is cyclic and conjugate to the companion matrix $C(f)$. In ([5], Thm. 7), Brawley and Teitloff prove that if $f \in \mathbb{F}_q[x]$ is monic of degree n and f is not square in $\mathbb{F}_q[x]$ then $C(f)$ is indeed similar to a symmetric matrix.

In conclusion, z is conjugate to some $h \in H$ which is inverted by t . Thus \bar{z} , and hence also \bar{y} , is strongly real as claimed. \square

4. UNITARY GROUPS

In this section we complete the proof of Theorem 1 for classical groups by proving

Theorem 4.1. *Let $\bar{G} = PSU_n(q)$ with $n \geq 3$ and $(n, q) \neq (3, 2)$. Then \bar{G} has involution width at most 4.*

After some preliminary material, we treat the cases of even and odd-dimensional unitary groups separately. Substantially more effort is required for the odd-dimensional case.

4.1. Preliminaries. Regard $G = SU_n(q) = \mathbf{G}^F$ where $\mathbf{G} = SL_n(\bar{\mathbb{F}}_q)$ and F is a Frobenius endomorphism as in Section 3.1. Explicitly, F is the standard Frobenius map that raises matrix entries to the q^{th} power, composed with the inverse-transpose map. Let Φ denote the root system of \mathbf{G} of type A_{n-1} . Then the maximal tori of G and its dual $G^* = PGU_n(q)$ correspond to F -conjugacy classes of the Weyl group $W = W(\Phi) \cong S_n$. These in turn correspond to S_n -orbits in the non-trivial coset of $\text{Aut}(\Phi) = S_n \times \{\pm 1\}$ ([33], 2.2). Such an orbit is given by a conjugacy class in S_n and therefore a partition (a_1, \dots, a_k) , of n . We denote the corresponding maximal torus by T_{a_1, \dots, a_k} . Recall that there exists a torus $T^* < G^*$ that is dual to T in the sense that the Frobenius actions on their character groups are mutually transpose ([33], 2.2). This duality gives a bijective correspondence between types of maximal tori in G and G^* . In particular, T_{a_1, \dots, a_k} and its dual denoted T_{a_1, \dots, a_k}^* have the same order

$$|T_{a_1, \dots, a_k}| = |T_{a_1, \dots, a_k}^*| = \frac{(q^{a_1} - (-1)^{a_1}) \dots (q^{a_k} - (-1)^{a_k})}{(q + 1)}. \quad (2)$$

Note that these orders, as well as those for all maximal tori of Lie type groups, can be calculated using the methods in [6].

Firstly we define what it means for two tori to be weakly orthogonal and explore how this affects the values of characters evaluated on such tori. This concept was first seen in [41] and then given a formal definition in [33].

Definition 4.2. We call two maximal tori $T_1 = \mathbf{T}_1^F$ and $T_2 = \mathbf{T}_2^F$ weakly orthogonal if

$$\mathbf{S}_1^{*F^*} \cap \mathbf{S}_2^{*F^*} = 1$$

for every pair $\mathbf{S}_1^*, \mathbf{S}_2^*$ of F^* -stable conjugates of T_1^*, T_2^* .

If x and y are regular semisimple elements of $G = \mathbf{G}^F$ with centralizers T_1 and T_2 respectively, we say x and y are weakly orthogonal if and only if T_1 and T_2 are weakly orthogonal.

For unitary groups there are known collections of weakly orthogonal tori pairs:

Proposition 4.3. ([33], 2.3.2). *Let $G = SU_n(q)$. For $0 \leq a \leq n-1$, the maximal tori T_n and $T_{1,a,n-1-a}$ are weakly orthogonal. If $2 \leq a \leq n-2$, the maximal tori $T_{1,n-1}$ and $T_{a,n-a}$ are weakly orthogonal.*

Here is a result concerning the vanishing of characters on weakly orthogonal elements.

Proposition 4.4. ([33], 2.2.2). *Let $G = SU_n(q)$. If x and y are weakly orthogonal regular semisimple elements of G , and χ is an irreducible character of G such that $\chi(x)\chi(y) \neq 0$, then χ is unipotent.*

It is thus useful when computing structure constants to consider weakly orthogonal elements as this allows for a reduction down to the unipotent characters, of which the most is known. Recall from Section 3.1 that the unipotent characters are exactly the elements of the Lusztig series $\mathcal{E}(G, (1))$.

For classical groups we have a combinatorial description for such characters ([8], 13.8). In particular, for a root system of type A_{n-1} , unipotent characters correspond to partitions of n .

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ be a partition of n and let $[\lambda]$ be the associated Young diagram. Denote the set of hooks of $[\lambda]$ by H , and the length of a hook $h \in H$ by $l(h)$. Also, define $a(\lambda) := \sum_{i < j} \min(\lambda_i, \lambda_j)$ and the integral polynomial

$$\rho_\lambda(x) = \frac{(x^n - 1)(x^{n-1} - 1) \dots (x - 1)}{\prod_{h \in H} (x^{l(h)} - 1)} \cdot x^{a(\lambda)}. \quad (3)$$

The degree of the corresponding unipotent character denoted $\chi_\lambda \in \text{Irr}(SU_n(q))$, is then given by $\chi_\lambda(1) = \pm \rho_\lambda(-q)$, where the sign is chosen such that the highest coefficient is positive. For example if $\lambda = (n-1, 1)$ where n is odd, then $\chi_\lambda(1) = q(q^{n-1} - 1)/(q + 1)$. Note also that the unipotent character of $SL_n(q)$ associated to λ has degree $\rho_\lambda(q)$.

Recall from Proposition 3.2 that a general character $\chi \in \text{Irr}(G)$ corresponds to a member of a Lusztig series $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$, where $s \in G^*$ is semisimple. Centralizers of semisimple elements are well understood in unitary groups ([7], Sec. 8) and we shall need to only consider the case where $C_{\mathbf{G}^*}(s)$ is connected. In particular, connected centralizers in $GU_n(q)$ of semisimple elements are direct products of factors of type $GL_{n_i}(q^{a_i})$ and $GU_{n_i}(q^{a_i})$. We then consider the image of these products in $G^* = PGU(q)$. A unipotent character of $C_{\mathbf{G}^*}(s)^{F^*}$ restricted to one of these components then corresponds to a partition $\lambda^i \vdash n_i$ as explained above. Hence letting these partitions form the entries of a multi-partition denoted $\bar{\lambda}$, it follows that a given $\chi \in \text{Irr}(G)$ is completely determined by s and $\bar{\lambda}$. We therefore adopt the notation $\chi = \chi_{s, \bar{\lambda}}$ to refer to this character. We however follow the usual notation for unipotent characters that we have already used above. That is, when $s = 1$ we write $\chi_{1, \bar{\lambda}} = \chi_\lambda$.

Note that the hooks contained in a diagram $[\lambda]$ match those of the diagram of the conjugate partition $[\lambda']$. Consequently, the degrees $\chi_\lambda(1)$ and $\chi_{\lambda'}(1)$ differ only by a scalar factor $q^{a(\lambda') - a(\lambda)}$.

We can further relate pairs of characters in $\text{Irr}(G)$ using the notion of Alvis-Curtis duality. For a reductive group \mathbf{G} , the Alvis-Curtis duality functor $D_{\mathbf{G}}$ sends any $\chi \in \text{Irr}(\mathbf{G}^F)$ to another irreducible character of \mathbf{G}^F up to sign ([13], 8.8). This duality is particularly important as we will often only have to find character values for one character in each pairing.

Lemma 4.5. *Let $g \in G$ be regular semisimple. Then $|\chi_{s,\bar{\lambda}}(g)| = |D_{\mathbf{G}}(\chi_{s,\bar{\lambda}})(g)|$.*

Proof. This follows from part of the proof of ([26], 2.3) which we will summarise here. By the definition of the functor $D_{\mathbf{G}}$ ([13], 8.8) it is clear that $D_{\mathbf{T}}(\theta) = \theta$ for any F -stable torus \mathbf{T} and $\theta \in \text{Irr}(\mathbf{T}^F)$. Recall that as g is regular semisimple, $C_{\mathbf{G}}(g)^F = T_1$ for some maximal torus $T_1 < G$. Applying this and ([13], Cor 8.16), we see that

$$|D_{\mathbf{G}}(\chi_{s,\bar{\lambda}})(g)| = |(D_{C_{\mathbf{G}}(g)} \circ \text{Res}_{C_{\mathbf{G}}(g)^F}^G)(\chi_{s,\bar{\lambda}})(g)| = |\chi_{s,\bar{\lambda}}(g)|.$$

□

Recent work of Guralnick, Larsen and Tiep [24] describes the Alvis-Curtis duality of unitary groups.

Lemma 4.6. ([24], 5.3). *Let $\tilde{G} = GU_n(q)$ and $\chi_{s,\bar{\lambda}} \in \text{Irr}(\tilde{G})$, where $s \in \tilde{G}^*$ is semisimple and $\bar{\lambda} = (\lambda^1, \lambda^2, \dots)$ is a multi-partition, as described above. Let $\bar{\mu} = ((\lambda^1)', (\lambda^2)', \dots)$. Then $D_{\tilde{G}}(\chi_{s,\bar{\lambda}}) = \chi_{s,\bar{\mu}}$.*

As we are working in $G = \mathbf{G}^F = SU_n(q)$, we consider the extension of characters to $\tilde{G} = \tilde{\mathbf{G}}^F = GU_n(q)$ before applying Lemmas 4.5 and 4.6. Recall that $G^* = PGU_n(q)$ and $\tilde{G}^* = GU_n(q)$, and let $\pi : \tilde{G}^* \rightarrow G^*$ denote the usual projection map. For a semisimple element $s \in G^*$, there exists $\tilde{s} \in \tilde{G}^*$ such that $\pi(\tilde{s}) = s$ and we have the Lusztig series $\mathcal{E}(G, (s)) \subseteq \text{Irr}(G)$ and $\mathcal{E}(\tilde{G}, (\tilde{s})) \subseteq \text{Irr}(\tilde{G})$. Furthermore,

$$\mathcal{E}(G, (s)) = \{\chi \in \text{Irr}(G) \mid \chi \text{ occurs in } \tilde{\chi}|_G \text{ for some } \tilde{\chi} \in \mathcal{E}(\tilde{G}, (\tilde{s}))\}.$$

To check if $\chi \in \text{Irr}(G)$ extends to a character of \tilde{G} we use the following result.

Lemma 4.7. ([29], 11.22). *Let H be a finite group and let $N \trianglelefteq G$ such that G/N is cyclic. Let $\chi \in \text{Irr}(N)$ be G -invariant. Then χ is extendible to G .*

To obtain information about the splitting of a character $\tilde{\chi} \in \mathcal{E}(\tilde{G}, (\tilde{s}))$ upon restriction from \tilde{G} to G , we study the centralisers $\tilde{H} = C_{\tilde{G}^*}(\tilde{s})$ and $H = C_{G^*}(s)$. Recall from Proposition 3.2 that $\psi_s(\tilde{\chi})$ is a unipotent character of \tilde{H}^F . Let Γ denote the stabiliser of $\psi_s(\tilde{\chi})$ under the action of $H^F/H^{\circ F}$. Then $\tilde{\chi}|_G$ splits into $|\Gamma|$ constituents. We will only consider Lusztig series $\mathcal{E}(G, (s))$ where $H = C_{G^*}(s)$ is connected and so the characters restrict irreducibly.

4.2. $G = SU_n(q)$, n even. In this section we complete the proof of Theorem 4.1 for even dimensions $n \geq 4$.

Proposition 4.8. *Let $G = SU_n(q)$, $n \geq 4$ even. There exist regular semisimple elements $x \in T_n$ and $y \in T_{1,1,n-2}$ such that \bar{x} and \bar{y} are strongly real in the simple group $\bar{G} = PSU_n(q)$.*

Proof. The maximal torus $T_n < G$ has order $\frac{q^n-1}{q+1}$ and is cyclic ([16], 4.5). Choose $x \in T_n$ of prime order r , where $r = \text{ppd}(q, n)$ if $n \equiv 0 \pmod{4}$ and $r = \text{ppd}(q, n/2)$ if $n \equiv 2 \pmod{4}$. Note in both cases r divides $q^{\frac{n}{2}} + (-1)^{\frac{n}{2}}$. Such primitive prime divisors exist (see Definition 3.4) Evidently x is regular as the order formula (2) prevents it from lying in any other maximal torus.

To show that x is strongly real in \bar{G} we look for containment in a strongly real subgroup. First note that G has an $SL_2(q^{\frac{n}{2}})$ subgroup containing x . Indeed, when $n \equiv 2 \pmod{4}$, $SU_2(q^{\frac{n}{2}}) \cong SL_2(q^{\frac{n}{2}})$ is a field extension subgroup of $SU_n(q)$, whereas if $n \equiv 0 \pmod{4}$ we have the embedding $SL_2(q^{\frac{n}{2}}) < SL_{\frac{n}{2}}(q^2) < SU_n(q)$.

In even characteristic, $PSL_2(q^{\frac{n}{2}}) = SL_2(q^{\frac{n}{2}})$ and this subgroup is strongly real by Theorem 2.1. In odd characteristic we can find $g \in SL_2(q^{\frac{n}{2}})$ such that $g^2 = -1$ and $x^g = -x^{-1}$. It follows that \bar{g} is an involution in \bar{G} and hence \bar{x} is strongly real.

A similar method is used for y where now we want to pick a regular element in a torus of size $|T_{1,1,n-2}| = (q^{n-2} - 1)(q + 1)$. We take y as the product of a regular element y_1 in the cyclic torus $T_{n-2} < SU_{n-2}(q)$ and $y_2 = \text{diag}(\mu, \mu^{-1}) \in T_{1,1} \leq SU_2(q)$ where $\mu \neq \mu^{-1}$ and $\mu\bar{\mu} = 1$. Here matrices in $SU_2(q)$ are written with respect to an orthonormal basis.

A regular element y_1 exists in $SU_{n-2}(q)$ exactly as we picked x above, and evidently $C_{SU_2(q)}(y_2) = T_{1,1}$. It follows that $C_{SU_n(q)}(y) = T_{1^2, n-2}$ and so y is regular. As above, there exists an element $g_1 \in SU_{n-2}(q)$ such that $g_1^2 = -1$ and $g_1 y_1 g_1^{-1} = -y_1^{-1}$. Let $g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU_2(q)$. Then it follows that $g := \text{diag}(g_1, g_2)$ is an involution in \bar{G} , conjugating \bar{y} to its inverse. Hence \bar{y} is strongly real. \square

For the remainder of this section, we fix the elements $x, y \in G = SU_n(q)$ ($n \geq 4$ and even) as in Proposition 4.8. We now calculate the normalised structure constants using the formula given in Theorem 2.7. Recall, we wish to show that for arbitrary $g \in G \setminus Z(G)$,

$$\sum_{\chi \in \text{Irr}(G)} \frac{\chi(x)\chi(y)\chi(g^{-1})}{\chi(1)} \neq 0.$$

In the following, let $\Phi_k(z)$ denote the usual k^{th} cyclotomic polynomial in the variable z .

Lemma 4.9. *Let $G = SU_n(q)$ where $n \geq 4$ is even. Suppose χ is an irreducible character of G such that $\chi(x)\chi(y) \neq 0$. Then χ is unipotent and corresponds to a partition $\lambda \in \{(n), (1^n), (n-1, 1), (2, 1^{n-2})\}$.*

Proof. By Proposition 4.3, x and y are weakly orthogonal. It therefore follows from Proposition 4.4 that χ is unipotent. We first consider unipotent characters such that $\chi_\lambda(x) \neq 0$. Recall that $\chi_\lambda(1) = |\rho_\lambda(-q)|$ where the polynomial $\rho_\lambda(x)$ is given by the formula (3), and x has prime order r , defined in the proof of Proposition 4.8. Clearly r divides $\rho_\lambda(-q)$ if and only if $\Phi_n(-q)$ divides $\rho_\lambda(-q)$. It follows that either $\rho_\lambda(-q)_r = 1$ or $\rho_\lambda(-q)_r = |G|_r$. But $\Phi_n(-q)$ divides $\rho_\lambda(-q)$ if and only if $[\lambda]$ has no hooks of length n . Hence $\chi_\lambda(x) \neq 0$ implies that $[\lambda]$ has an n -hook and so the partition must be of the form $\lambda = (k, 1^{n-k})$ for some k .

The case for characters not vanishing on y is similar as the order of y is divisible by a primitive prime divisor s , of $\Phi_{n-2}(-q)$. If $n \geq 6$ then as above, either $\rho(-q)_s = 1$ or $\rho(-q)_s = |G|_s$. Hence if $\chi_\lambda(y) \neq 0$, then $[\lambda]$ contains an $(n-2)$ -hook. In summary, if $\chi_\lambda(x)\chi_\lambda(y) \neq 0$ then $[\lambda]$ contains both an n -hook and an $(n-2)$ -hook. This is only possible if $\lambda \in \{(n), (1^n), (n-1, 1), (2, 1^{n-2})\}$.

The result for $n = 4$ follows without the need to consider vanishing on y as the partitions listed in the statement are exactly those containing a 4-hook. \square

Proposition 4.10. *Let $G = SU_n(q)$ where $n \geq 4$ is even and $(n, q) \notin \{(4, 2), (4, 3), (6, 2)\}$. Let $g \in G$ be a non-semisimple element. Then $g \in x^G \cdot y^G$.*

Proof. We show that the normalised structure constant is non-zero by using the formula given in Theorem 2.7. By Lemma 4.9 we thus far have a reduction to

$$\kappa(x^G, y^G, g^G) = \sum_{\chi_\lambda} \frac{\chi_\lambda(x)\chi_\lambda(y)\chi_\lambda(g^{-1})}{\chi_\lambda(1)} \quad (4)$$

where the sum is over $\lambda \in \{(n), (1^n), (n-1, 1), (2, 1^{n-2})\}$.

Note that χ_n is the trivial character and χ_{1^n} is the Steinberg character St which takes absolute value 1 on regular semisimple elements ([8], 6.4.7). Also note the character degrees:

$$St(1) = q^{\frac{n}{2}(n-1)}, \quad \chi_{(n-1,1)}(1) = \frac{q(q^{n-1}+1)}{q+1}, \quad \chi_{(2,1^{n-2})}(1) = \frac{q^{\frac{1}{2}(n-2)(n-1)}(q^{n-1}+1)}{q+1}.$$

Now $\chi_{(n-1,1)}$ is one of the so called irreducible Weil characters, some characterisations of which are given by Tiep and Zalesskii in [53]. Fix a generating element γ of $\mathbb{F}_{q^2}^*$ and let $\delta = \gamma^{q-1}$. Then by ([53], 4.1) we can compute the character values using the formula

$$\chi_{(n-1,1)}(g) = \frac{(-1)^n}{q+1} \sum_{l=0}^q (-q)^{\dim \text{Ker}(g-\delta^{-l})}. \quad (5)$$

By the choice of order, it follows that x has no eigenvalues contained in $\langle \delta \rangle$ and hence $\chi_{(n-1,1)}(x) = 1$. Similarly, $\chi_{(n-1,1)}(y) = -1$ as there exist two 1-dimensional eigenspaces for the eigenvalues μ and μ^{-1} .

From ([53], Sec. 4) we see that $\chi_{(n-1,1)} \in \mathcal{E}(G, (1))$ is the restriction of the unipotent character $\tilde{\chi}_{(n-1,1)} \in \text{Irr}(\tilde{G})$ and it follows from Lemmas 4.5 and 4.6 that $|\chi_{(n-1,1)}(x)| = |\tilde{\chi}_{(n-1,1)}(x)| = |\tilde{\chi}_{(2,1^{n-2})}(x)|$. Then, as $C_{\mathbf{G}^*}(1) = \mathbf{G}^*$ is connected, it follows from Section 4.1 that $\tilde{\chi}_{(2,1^{n-2})}|_G = \chi_{(2,1^{n-2})}$. Hence $|\chi_{(n-1,1)}(x)| = |\chi_{(2,1^{n-2})}(x)| = 1$. It follows in an identical manner that $|\chi_{(2,1^{n-2})}(y)| = 1$.

The value of the Steinberg character is given by ([8], 6.4.7) : $St(z) = \pm |C_G(z)|_p$ if z is semisimple and zero otherwise. Hence the Steinberg summand vanishes in our consideration of (4). Combining these results, we simplify (4) to

$$\kappa(x^G, y^G, g^G) = 1 + \frac{-(q+1)\chi_{(n-1,1)}(g^{-1})}{q(q^{n-1}+1)} \pm \frac{\chi_{(2,1^{n-2})}(g^{-1})(q+1)}{q^{\frac{1}{2}(n-2)(n-1)}(q^{n-1}+1)}.$$

As g is non-scalar it cannot have an eigenvalue in $\langle \delta \rangle$ of multiplicity n and thus $|\chi_{(n-1,1)}(g)| \leq \frac{1}{q+1}|(-q)^{n-1} - 1|$. We do not have a method for computing general character values of $\chi_{(2,1^{n-2})}$ but it is enough to use the trivial bound $|\chi_{(2,1^{n-2})}(g)| \leq |C_G(g)|^{\frac{1}{2}}$. Centralizer orders can be easily computed using the details in ([36], 7.1 and [7], Sec. 8) and this bound is maximal when g is unipotent with Jordan block structure $J_2 \oplus J_1^{n-2}$. Excluding a small number of exceptions when n and q are small, we find that $\frac{|\chi_{(2,1^{n-2})}(g^{-1})|}{\chi_{(2,1^{n-2})}(1)} < \frac{1}{q^2}$. In any case $\frac{|\chi_{(2,1^{n-2})}(g^{-1})|}{\chi_{(2,1^{n-2})}(1)} < \frac{q-1}{q}$ and hence

$$\frac{(q+1)|\chi_{(n-1,1)}(g)|}{q(q^{n-1}+1)} + \frac{|\chi_{(2,1^{n-2})}(g^{-1})(q+1)|}{q^{\frac{1}{2}(n-2)(n-1)}(q^{n-1}+1)} < \frac{1}{q} + \frac{q-1}{q} = 1.$$

We conclude that $\kappa(x^G, y^G, g^G) \neq 0$ and thus $g \in x^G \cdot y^G$. \square

Proof of Theorem 4.1 for n even:

Let $G = SU_n(q)$ where $n \geq 4$ is even. If $(n, q) \in \{(4, 2), (4, 3), (6, 2)\}$ then the result can be checked using GAP [19] so we assume this is not the case. Let $g \in G$ be a semisimple element. Then by Theorem 3.5, it follows that $\bar{g} \in x^{\bar{G}}y^{\bar{G}}$. Similarly, $\bar{g} \in x^{\bar{G}}y^{\bar{G}}$ for all non-semisimple elements $g \in G$ by Proposition 4.10. As \bar{x} and \bar{y} are strongly real by Proposition 4.8, it follows that $\text{iw}(\bar{G}) \leq 4$ and Theorem 4.1 is proved for even $n \geq 4$.

4.3. $G = SU_n(q)$, n odd. This section completes our work on the classical groups by considering the simple unitary groups of odd dimension. More precisely we complete the proof of Theorem 4.1 for odd $n \geq 3$. Note that the initial cases $PSU_3(q)$ and $PSU_5(q)$ can be dealt with easily using the computer package CHEVIE ([21]). We do however begin with a slightly more detailed treatment of $SU_3(q)$ as knowledge of the involution width here will be of particular use later in the analysis of exceptional Lie-type groups.

Lemma 4.11. *Let $G = SU_3(q)$ where $q \neq 2$. Then G has involution width at most 4.*

Proof. Calculation of structure constants is straightforward in $SU_3(q)$ as the full character table is known ([48]). Furthermore, products of conjugacy classes in $SU_3(q)$ have been examined in the work of Orekovkov [45].

Firstly we note that $SU_3(q)$ contains a unique class of involutions regardless of characteristic. If 3 does not divide $q + 1$, then $PSU_3(q) = SU_3(q)$ and $\text{iw}(G) \leq 4$ by ([45], 1.9). Hence assume now that 3 divides $q + 1$ and $q \neq 2$. Theorem 1.3 of [45] details when exactly the identity element is contained in a product of conjugacy classes, and denoting our class of involutions by A , we find that for an arbitrary class $1 \neq C \subset G$ either $1 \in A^3C$ or $1 \in A^4C$. It follows that the involution width of $SU_3(q)$ is at most 4 for $q \neq 2$. \square

Lemma 4.12. *$PSU_5(q)$ has involution width at most 4.*

Proof. The generic character table of $PGU_5(q)$ is available in the computer package CHEVIE [21]. These tables can be accessed with Maple ([43]) and we check the relevant structure constants computationally. \square

The remainder of this section will be devoted to proving Theorem 4.1 where $n \geq 7$ is odd. Throughout, let $G = SU_n(q)$.

The proof of this result will follow the same method as the case when n is even. That is, we first pick $x, y \in G$ such that the projections of these elements are strongly real in \overline{G} and then show that $G \setminus Z(G) \subset x^G \cdot y^G$. However unlike before, when n is odd we do not pick x and y from weakly orthogonal tori. Thus when computing structure constants we now have non-unipotent characters to contend with, which makes things much more complicated, as we shall see.

Proposition 4.13. *There exist regular semisimple elements $x \in T_{1,n-1}$ and $y \in T_{1^3,n-3}$ such that x and y are strongly real in G .*

Proof. Firstly note that the torus $T_{1,n-1}$ of order $q^{n-1} - 1$ is cyclic and we can pick $x \in T_{1,n-1}$ of order r , where $r = \text{ppd}(q, n-1)$ when $n-1 \equiv 0 \pmod{4}$ and $r = \text{ppd}(q, \frac{n-1}{2})$ when $n-1 \equiv 2 \pmod{4}$. These primes exist by Zsigmondy's Theorem. Now x has minimal polynomial of the form $p(t) = f(t)(t-1)$ where $f(t)$ is a degree $n-1$ polynomial, irreducible over \mathbb{F}_{q^2} . Furthermore, $f(t)$ is self-reciprocal, meaning that if $\alpha_1, \dots, \alpha_{n-1} \in \overline{\mathbb{F}}_q$ are the roots of f , then $\prod(t - \alpha_i) = f(t) = \prod(t - \alpha_i^{-1})$. Clearly the elementary divisor $(t-1)$ is also self-reciprocal and it follows that x is real in $SU_n(q)$. It then follows from ([50], 7.1) and ([20], 2.2 and 2.4) that x is strongly real in $SU_n(q)$.

A similar method is used for y where we now pick a regular semisimple element in the torus $T_{1^3,n-3}$ of size $|T_{1^3,n-3}| = (q^{n-3} - 1)(q+1)^2$. We take y as the product of a regular element y_1 in the cyclic torus $T_{n-3} < SU_{n-3}(q)$ and $y_2 = \text{diag}(\mu, \mu^{-1}, 1) \in T_{1^3} < SU_3(q)$. In particular y_1 has order r' where $r' = \text{ppd}(q, n-3)$ when $n-3 \equiv 0 \pmod{4}$ and $r' = \text{ppd}(q, \frac{n-3}{2})$ when $n-3 \equiv 2 \pmod{4}$. Furthermore, if $\langle \delta \rangle \cong C_{q+1} \subset \mathbb{F}_{q^2}^*$ and $\epsilon = e^{2\pi i/(q+1)}$, then we choose $\mu = \delta^a$ such that $|1 + 2\text{Re}(\epsilon^a)| \leq 1$. This is always possible and we can also ensure that $\mu \notin \{1, -1\}$. In fact if $q \equiv -1 \pmod{3}$ then we pick μ such that $1 + 2\text{Re}(\epsilon^a) = 0$. This will be important for later calculations of structure constants.

Clearly y_2 has the torus T_{1^3} as its centralizer in $SU_3(q)$ and it follows that $C_{SU_n(q)}(y) = T_{1^3,n-3}$ and so y is regular. Note that y has self-reciprocal minimal polynomial $p(t) = f(t)(t-\mu)(t-\mu^{-1})(t-1)$ such that $f(t)$ is of degree $n-3$, self-reciprocal, and irreducible over \mathbb{F}_{q^2} . Hence it follows as above that y is strongly real in $SU_n(q)$. \square

As we saw in Section 4.2, the bulk of the work is now to show that the product $x^G \cdot y^G$ contains all non-semisimple elements of G . Unlike before however, the two tori $T_{1,n-1}$ and $T_{1^3,n-3}$ are not weakly orthogonal. Hence when computing the structure constant it is

possible that $\chi(x)\chi(y) \neq 0$ for a far greater range of characters $\chi \in \text{Irr}(G)$. Exactly when a character may be non-vanishing is detailed below.

Proposition 4.14. *Suppose $\chi \in \text{Irr}(G)$ such that $\chi(x)\chi(y) \neq 0$. Then χ is one of the following characters.*

(1) *Unipotent characters:*

$$1_G, St, \chi_{(n-3,2,1)}, \chi_{(3,2,1^{n-5})}.$$

(2) *Non-unipotent characters:*

$$\chi_{s^t, (n-1)}, \chi_{s^t, (1^{n-1})}, \chi_{s^t, (n-2,1)}, \chi_{s^t, (2,1^{n-3})}.$$

Here $s = \text{diag}(1, \dots, 1, \delta)$ where $\langle \delta \rangle \cong C_{q+1} \subset \mathbb{F}_{q^2}$, and $1 \leq t \leq q$. Also $C_{G^*}(s) \cong GU_{n-1}(q)$.

Proof. Applying Lemma 3.1 to the elements x and y , it follows that $\chi \in \mathcal{E}(G, (s))$ where $s \in T_{1, n-1}^* \cap T_{1^3, n-3}^*$. This intersection consists of images in G^* of elements of the form $s = \text{diag}(1, \dots, 1, \delta^t)$ where $0 \leq t \leq q$. First consider the case of possible unipotent characters, that is when $t = 0$ and so $s = 1$. We proceed as in Lemma 4.9, proving that only the 4 unipotent characters χ_λ listed in the conclusion are not of r - or r' -defect zero (where r, r' are as defined in the proof of Proposition 4.13) and then apply Theorem 3.3.

Firstly assume that χ_λ is a unipotent character such that $\chi_\lambda(x) \neq 0$. From the degree formula (3), it is clear that r divides $\chi_\lambda(1) = \rho_\lambda(-q)$ if and only if $\Phi_{n-1}(-q)$ divides $\rho_\lambda(-q)$. It follows that either $\rho_\lambda(-q)_r = 1$ or $\rho_\lambda(-q)_r = |G|_r$. Then note that $\Phi_{n-1}(-q)$ divides $\rho_\lambda(-q)$ if and only if $[\lambda]$ has no hooks of length $n-1$. Hence $\chi_\lambda(x) \neq 0$ implies that $[\lambda]$ does indeed have an $(n-1)$ -hook and so the partition must be of the form $\lambda = (n), (1^n)$ or $(k, 2, 1^{n-k-2})$ for some k .

The case for characters that are possibly non-vanishing on y is similar as the order of y is divisible by a primitive prime divisor factor r' , of $\Phi_{n-3}(-q)$. As $n \geq 7$, either $\rho_\lambda(-q)_{r'} = 1$ or $\rho_\lambda(-q)_{r'} = |G|_{r'}$. Hence if $\chi_\lambda(y) \neq 0$, then $[\lambda]$ contains an $(n-3)$ -hook. In summary, if $\chi_\lambda(x)\chi_\lambda(y) \neq 0$ then $[\lambda]$ contains both an $(n-1)$ -hook and an $(n-3)$ -hook. Namely $\lambda \in \{(n), (1^n), (n-3, 2, 1), (3, 2, 1^{n-5})\}$.

To conclude, consider non-unipotent characters $\chi \in \mathcal{E}(G, (s^t))$ where $1 \leq t \leq q$. Clearly $C = C_{G^*}(s) \cong GU_{n-1}(q)$ and thus $\chi = \chi_{s^t, \lambda}$ where $\lambda \vdash n-1$. As explained above, $\chi(x)\chi(y) \neq 0$ implies that $[\lambda]$ contains both an $(n-1)$ - and $(n-3)$ -hook. Hence $\lambda \in \{(n-1), (1^{n-1}), (n-2, 1), (2, 1^{n-3})\}$ and we have the result. \square

For the remainder of this section, denote the set of irreducible characters listed in (1) and (2) of Proposition 4.14 by $X \subset \text{Irr}(G)$.

4.3.1. Values of characters in X . Before computing the structure constants in full, we compute the values of each character $\chi \in X$ on the elements x and y defined in Proposition 4.13. Firstly, we have 1_G and $St_G \in X$. These characters are well understood and we have seen earlier that explicit formulae exist.

Next consider $\chi_{s^t, (n-1)}$ and the dual $\chi_{s^t, (1^{n-1})}$. For ease of notation denote $F = \mathbb{F}_{q^2}$. The reducible Weil character $\zeta_{n,q}$ of $GU_n(q)$ is defined by the formula

$$\zeta_{n,q}(g) = (-1)^n (-q)^{\dim \text{Ker}_F(g-1)}. \quad (6)$$

This character is well studied in the literature ([53], Sec. 4); in particular it has the following decomposition when restricted to $G = SU_n(q)$:

$$\zeta_{n,q}|_G = \chi_{(n-1,1)} + \sum_{t=1}^q \chi_{s^t, (n-1)}. \quad (7)$$

Furthermore explicit formulae for the values of each $\chi_{s^t, (n-1)}$ are known.

Lemma 4.15. ([53], 4.1). *Let $\epsilon = e^{2\pi i/(q+1)}$ and $\langle \delta \rangle \cong C_{q+1} \subset \mathbb{F}_{q^2}$. Then for $g \in G = SU_n(q)$,*

$$\chi_{s^t, (n-1)}(g) = \frac{(-1)^n}{q+1} \sum_{l=0}^q \epsilon^{-tl} (-q)^{\dim \text{Ker}_F(g-\delta^{-l})}.$$

Far less is known about the dual characters $\chi_{s^t, (1^{n-1})}$. We can however apply Lemmas 4.5 and 4.15 to find $|\chi_{s^t, (1^{n-1})}(x)|$ and $|\chi_{s^t, (1^{n-1})}(y)|$.

The second family of non-unipotent characters $\chi_{s^t, (n-2, 1)}$ has also been studied in the more recent literature. Now ([34], 6.6) describes any character $\chi \in \text{Irr}(SU_n(q))$ of degree less than the order of q^{3n-9} by associating χ to an $\alpha \in \text{Irr}(GU_2(q))$. In particular we see that for a fixed t , $\chi_{s^t, (n-2, 1)}$ has labeling D_α° where $\alpha = \chi_{q-1}^{(q+1, u)}$ for some $u \in \{1, \dots, q\}$. Here the labeling of $\text{Irr}(GU_2(q))$ is that used by Ennola ([15]). We then have by ([34], 5.5) that for $g \in SU_n(q)$,

$$\chi_{s^t, (n-2, 1)}(g) = D_\alpha^\circ(g) = \frac{1}{|GU_2(q)|} \sum_{z \in GU_2(q)} \overline{\alpha(z)} \omega(zg) \quad (8)$$

where $\omega(h) = \zeta_{2n}(h) = (-q)^{\dim \text{Ker}_F(h-1)}$ is the reducible Weil character of $GU_{2n}(q)$ of degree q^{2n} and $zg \in GU_2(q) \otimes GU_n(q) \subseteq GU_{2n}(q)$.

For the dual characters $\chi_{s^t, (2, 1^{n-3})}$, we will again see that an application of Lemma 4.5 or the trivial bound, $|\chi(g)| \leq |C_G(g)|^{\frac{1}{2}}$, will be sufficient for calculating structure constants.

Lemma 4.16. *Let $x, y \in SU_n(q)$ be the regular semisimple elements defined in Proposition 4.13. Recall that y has eigenvalue μ chosen such that $\mu = \delta^a$. Then*

$$\begin{aligned} \chi_{s^t, (n-1)}(x) &= \chi_{s^t, (n-2, 1)}(x) = 1, \\ \chi_{s^t, (n-1)}(y) &= \chi_{s^t, (n-2, 1)}(y) = 1 + 2\text{Re}(\epsilon^a), \\ |\chi_{s^t, (1^{n-1})}(x)| &= |\chi_{s^t, (2, 1^{n-2})}(x)| = 1, \\ |\chi_{s^t, (1^{n-1})}(y)| &= |\chi_{s^t, (2, 1^{n-2})}(y)| = |1 + 2\text{Re}(\epsilon^a)|. \end{aligned}$$

Proof. The values of $\chi_{s^t, (n-1)}$ follow from Lemma 4.15. The values of $\alpha = \chi_{q-1}^{(q+1, u)} \in \text{Irr}(GU_2(q))$ are given in full in ([15]) and we then use (8) to evaluate $D_\alpha^\circ = \chi_{s^t, (n-2, 1)}$.

To evaluate the characters $\chi_{s^t, (1^{n-1})}$ and $\chi_{s^t, (2, 1^{n-2})}$ we follow the method given in the proof of Proposition 4.10: from ([53], Sec. 4) we see that $\chi_{s^t, (n-1)}$ is the restriction of the irreducible Weil character $\tilde{\chi}_{\tilde{s}^t, (n-1)} \in \text{Irr}(\tilde{G})$ where \tilde{s} is a fixed preimage of $s \in G^*$. It follows from Lemmas 4.5 and 4.6 that $|\chi_{s^t, (n-1)}(x)| = |\tilde{\chi}_{\tilde{s}^t, (n-1)}(x)| = |\tilde{\chi}_{\tilde{s}^t, (1^{n-1})}(x)|$. Then, as $C_{G^*}(s^t) \cong GL_{n-1}(\overline{\mathbb{F}}_q)$ is connected, it follows from Section 4.1 that $\tilde{\chi}_{\tilde{s}^t, (1^{n-1})}|_G = \chi_{s^t, (1^{n-1})}$. Hence $|\chi_{s^t, (n-1)}(x)| = |\chi_{s^t, (1^{n-1})}(x)| = 1$. The value of $|\chi_{s^t, (1^{n-1})}(y)|$ follows identically.

To evaluate $|\chi_{s^t, (2, 1^{n-2})}(x)|$, first note from formula (8) that $\chi_{s^t, (n-2, 1)}$ is \tilde{G} -invariant. Hence as \tilde{G}/G is cyclic, $\chi_{s^t, (n-2, 1)}$ extends to a character of \tilde{G} by Lemma 4.7. From Sec 4.1 such extensions lie in Lusztig series $\mathcal{E}(\tilde{G}, \tilde{s}z)$, where \tilde{s} is a fixed preimage of $s^t \in G^*$ and $z \in Z(G)$. Note that if $\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s}z)$ restricts irreducibly to $\chi_{s^t, (n-2, 1)}$, then $\tilde{\chi}(1)_p = \chi_{s^t, (n-2, 1)}(1)_p = q$. Checking degrees of the unipotent characters of $C_{\tilde{G}}(\tilde{s}z) \cong GU_{n-1}(q) \times GU_1(q)$, it follows that $\tilde{\chi} = \tilde{\chi}_{\tilde{s}z, (n-2, 1)}$. Noting again that $C_{G^*}(s^t)$ is connected, the proof now follows exactly as above, by considering the Alvis-Curtis dual of $\tilde{\chi}$. \square

Note that when we apply these values later in the calculation of structure constants, we will only require the bound $|1 + 2\operatorname{Re}(\epsilon^a)| \leq 1$ given in the proof of Proposition 4.13. However recall that when $q = 2$, y is chosen such that $1 + 2\operatorname{Re}(\epsilon^a) = 0$.

Lastly we want to estimate values of the unipotent character $\chi_{(n-3,2,1)}$. We use the theory of dual pairs to extend the work in ([34], Sec. 6.1).

Let $G = SU_n(q)$ with $n \geq 7$ and odd and let $S := GU_3(q)$. Take $G = SU(W)$ where $W = \langle v_1, \dots, v_n \rangle_{\mathbb{F}_{q^2}}$ is endowed with a Hermitian form (\cdot, \cdot) , with Gram matrix $\operatorname{diag}(1, \dots, 1)$ with respect to the basis v_1, \dots, v_n . Similarly we view S as $GU(U)$ where $U = \langle e_1, e_2, e_3 \rangle_{\mathbb{F}_{q^2}}$ is endowed with Hermitian form (\cdot, \cdot) , with Gram matrix $\operatorname{diag}(1, 1, 1)$ in the basis e_1, e_2, e_3 . Next we consider $V = U \otimes W$ with the Hermitian form (\cdot, \cdot) defined via $(u \otimes w, u' \otimes w') = (u, u') \cdot (w, w')$ for $u, u' \in U$ and $w, w' \in W$. The action of $S \times G$ on V then induces a natural homomorphism $S \times G \rightarrow \Gamma := GU(V) = GU_{3n}(q)$.

Let $\omega = \zeta_{3n,q}$ denote the reducible Weil character of Γ , as defined by (6). It is by studying the restriction of ω to $S \times G$ that we will find the character $\chi_{(n-3,2,1)}$.

Lemma 4.17. *Let $n \geq 7$. Then*

$$(\omega|_G, \omega|_G)_G = (q+1)(q^3+1)(q^5+1).$$

Further $(\omega|_G, 1_G)_G = 0$ and $(\omega|_G, \zeta_{n,q})_G = (q+1)(q^3+1)$.

Proof. Let A be the matrix of $g \in G$ in the basis v_1, \dots, v_n of W . Then g has matrix $\operatorname{diag}(A, A, A)$ in the basis $\{e_1 \otimes v_i, e_2 \otimes v_i, e_3 \otimes v_i\}$ of V . It also follows that $\omega|_G = (\zeta_{n,q})^3$ where $\zeta_{n,q}$ denotes the reducible Weil character of $GU_n(q)$ as in (6). Furthermore

$$\begin{aligned} (\omega|_G, \omega|_G)_G &= \frac{1}{|G|} \sum_{A \in G} \omega(A)^2 \\ &= \frac{1}{|G|} \sum_{A \in G} \operatorname{Fix}(A)^3, \end{aligned}$$

where $\operatorname{Fix}(A)$ denotes the number of fixed points under the action on the natural module W . But this is exactly the number of G -orbits on $W \times W \times W$. Using Witt's lemma and the assumptions on n we find that the number of G -orbits is exactly $(q+1)(q^3+1)(q^5+1)$. Next note that $(\omega|_G, 1_G)_G = (\zeta_{n,q}^2, \zeta_{n,q})_G$ and similarly $(\omega|_G, \zeta_{n,q})_G = (\zeta_{n,q}^2, \zeta_{n,q}^2)_G$. The decomposition of $\zeta_{n,q}^2$ is studied in ([34], 6.1) and we can check that indeed $(\omega|_G, 1_G)_G = 0$ and $(\omega|_G, \zeta_{n,q})_G = (q+1)(q^3+1)$. \square

Proposition 4.18. *Let $S = GU_3(q)$ and $G = SU_n(q)$ with $n \geq 7$ odd. Then the restriction $\zeta_{3n,q}|_{S \times G}$ decomposes as $\sum_{\alpha \in \operatorname{Irr}(S)} \alpha \otimes D_\alpha$ where D_α is a character of G . Define $a_\alpha = (D_\alpha, \chi_{(n-1,1)})_G$, $b_\alpha^t = (D_\alpha, \chi_{s^t, (n-1)})_G$, for $1 \leq t \leq q$, and*

$$D_\alpha^\circ = D_\alpha - a_\alpha \chi_{(n-1,1)} - \sum_{t=1}^q b_\alpha^t \chi_{s^t, (n-1)}.$$

Then the characters D_α° of $G = SU_n(q)$ are all irreducible and distinct.

Proof. Applying ([34], 5.5) to the character $\omega = \zeta_{3n,q}$ gives the decomposition $\omega|_{S \times G} = \sum_{\alpha \in \operatorname{Irr}(S)} \alpha \otimes D_\alpha$ such that

$$D_\alpha(g) = \frac{1}{|S|} \sum_{x \in S} \overline{\alpha(x)} \omega(xg). \quad (9)$$

By definition,

$$\omega|_G = \sum_{\alpha \in \operatorname{Irr}(S)} \alpha(1) \left(a_\alpha \chi_{(n-1,1)} + \sum_{t=1}^q b_\alpha^t \chi_{s^t, (n-1)} \right) + \sum_{\alpha \in \operatorname{Irr}(S)} \alpha(1) D_\alpha^\circ$$

and it follows from Lemma 4.17 and (7) that

$$\sum_{\alpha \in \text{Irr}(S)} \alpha(1) \left(a_\alpha + \sum_{t=1}^q b_\alpha^t \right) = (q^3 + 1)(q + 1). \quad (10)$$

For ease of notation, define $\sum_{\alpha \in \text{Irr}(S)} \alpha(1) a_\alpha = a$ and $\sum_{\alpha \in \text{Irr}(S)} \alpha(1) b_\alpha^t = b_t$. It also follows from Lemma 4.17 that

$$\sum_{\alpha \in \text{Irr}(S)} \alpha(1)^2 = |S| = q^3(q^3 + 1)(q^2 - 1)(q + 1) = (\omega|_G, \omega|_G)_G - (q + 1)(q^3 + 1)^2.$$

Hence

$$\sum_{\alpha \in \text{Irr}(S)} \alpha(1)^2 = \left(\sum_{\alpha \in \text{Irr}(S)} \alpha(1) D_\alpha^\circ, \sum_{\alpha \in \text{Irr}(S)} \alpha(1) D_\alpha^\circ \right)_G + (a^2 + \sum_{t=1}^q b_t^2) - (q + 1)(q^3 + 1)^2.$$

Applying the Cauchy-Schwarz inequality to (10) yields

$$a^2 + \sum_{t=1}^q b_t^2 \geq (q + 1)(q^3 + 1)^2.$$

Thus if each D_α° , $\alpha \in \text{Irr}(S)$ has positive degree, it will follow that the characters are irreducible and distinct. The character table of S is known and we follow the notation of [15]. In particular there are 8 families of irreducible characters of degrees 1, $q^2 - q$, q^3 , $q^2 - q + 1$, $q(q^2 - q + 1)$, $(q - 1)(q^2 - q + 1)$, $q^3 + 1$ and $(q + 1)(q^2 - 1)$.

We compute the corresponding $D_\alpha(1)$ using (9) and it is a straightforward check that $D_\alpha(1) > q^n(q^3 + 1)(q + 1)/\alpha(1)$ for all $\alpha \in \text{Irr}(S)$. Furthermore, by the definition of D_α° ,

$$D_\alpha(1) - D_\alpha^\circ(1) \leq (\omega|_G, \zeta_{n,q})_G \zeta_{n,q}(1)/\alpha(1) = q^n(q + 1)(q^3 + 1)/\alpha(1). \quad (11)$$

Hence $D_\alpha^\circ(1) > 0$ for all $\alpha \in \text{Irr}(S)$ and the proof is finished. \square

Remark. By the proof of Proposition 4.18, $a^2 + \sum_{t=1}^q b_t^2 = (q + 1)(q^3 + 1)^2$ and it follows from Cauchy-Schwarz that $a = b_t = q^3 + 1$ for $1 \leq t \leq q$. Hence

$$\omega|_G = (q^3 + 1)\zeta_{n,q} + \sum_{\alpha \in \text{Irr}(S)} \alpha(1) D_\alpha^\circ.$$

The degrees $D_\alpha(1)$ are listed in Table 1. The notation for characters $\alpha \in \text{Irr}(GU_3(q))$ is taken from [15].

Recall that we wish to find a formula for the values of the unipotent character $\chi_{(n-3,2,1)}$. From Table 1, we see that there is precisely one character D_α of the correct degree. Specifically, when $\alpha = \chi_{q^2-q}^{(q+1)}$,

$$D_\alpha(1) = \frac{q^4(q^n + 1)(q^{n-2} + 1)(q^{n-4} + 1)}{(q^3 + 1)(q + 1)^2} = \chi_{(n-3,2,1)}(1).$$

Proposition 4.19. *Let $\alpha = \chi_{q^2-q}^{(q+1)} \in \text{Irr}(GU_3(q))$. Then $D_\alpha = D_\alpha^\circ = \chi_{(n-3,2,1)}$.*

From this we can then calculate values of $\chi_{(n-3,2,1)}$ using the known character table of $GU_3(q)$ and the formula (9). Define

$$d := \frac{q^n(q + 1)(q^3 + 1)}{q(q - 1)}.$$

To prove the proposition we first note from (11) in the proof of Proposition 4.18, that

$$D_\alpha(1) - D_\alpha^\circ(1) \leq d.$$

TABLE 1. Degrees of D_α for $G = SU_n(q)$

$\alpha \in \text{Irr}(GU_3(q))$	$D_\alpha(1)$
$\chi_1^{(q+1)}$	$\frac{q^3(q^n+1)(q^{n-1}-1)(q^{n-5}-1)}{(q^3+1)(q^2-1)(q+1)} + \frac{q(q^{n-1}-1)}{(q+1)}$
$\chi_1^{(t)}, 1 \leq t \leq q$	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^3+1)(q^2-1)(q+1)}$
$\chi_{q^3}^{(q+1)}$	$\frac{q^6(q^{n-1}-1)(q^{n-2}+1)(q^{n-4}+1)}{(q^3+1)(q^2-1)(q+1)} + \frac{q(q^{n-1}-1)}{(q+1)}$
$\chi_{q^3}^{(t)}, 1 \leq t \leq q$	$\frac{q^3(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^3+1)(q^2-1)(q+1)}$
$\chi_{q^2-q}^{(q+1)}$	$\frac{q^4(q^n+1)(q^{n-2}+1)(q^{n-4}+1)}{(q^3+1)(q+1)^2}$
$\chi_{q^2-q}^{(t)}, 1 \leq t \leq q$	$\frac{q(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^3+1)(q+1)^2}$
$\chi_{q^2-q+1}^{(t,q+1)}, 1 \leq t \leq q$	$\frac{q^2(q^n+1)(q^{n-1}-1)(q^{n-4}+1)}{(q^2-1)(q+1)^2} + \frac{(q^n+1)}{(q+1)}$
$\chi_{q^2-q+1}^{(q+1,u)}, 1 \leq u \leq q$	$\frac{q(q^n+1)(q^{n-1}-1)(q^{n-3}-1)}{(q^2-1)(q+1)^2}$
$\chi_{q^2-q+1}^{(t,u)}, 1 \leq t \neq u \leq q$	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^2-1)(q+1)^2}$
$\chi_{q(q^2-q+1)}^{(t,q+1)}, 1 \leq t \leq q$	$\frac{q^3(q^n+1)(q^{n-1}-1)(q^{n-3}-1)}{(q^2-1)(q+1)^2} + \frac{(q^n+1)}{(q+1)}$
$\chi_{q(q^2-q+1)}^{(q+1,u)}, 1 \leq u \leq q$	$\frac{q^2(q^n+1)(q^{n-1}-1)(q^{n-3}-1)}{(q^2-1)(q+1)^2}$
$\chi_{q(q^2-q+1)}^{(t,u)}, 1 \leq t \neq u \leq q$	$\frac{q(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^2-1)(q+1)^2}$
$\chi_{(q-1)(q^2-q+1)}^{(t,u,q+1)}, 1 \leq t < u < q+1$	$\frac{q(q^n+1)(q^{n-1}-1)(q^{n-3}-1)}{(q+1)^3}$
$\chi_{(q-1)(q^2-q+1)}^{(t,u,v)}, 1 \leq t < u < v < q+1$	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q+1)^3}$
$\chi_{q^3+1}^{(q+1,u)}, 1 \leq u \leq q^2-2$	$\frac{q(q^n+1)(q^{n-1}-1)(q^{n-3}-1)}{(q^2-1)(q+1)}$
$u \not\equiv 0 \pmod{q-1}$	
$\chi_{q^3+1}^{(t,u)}, 1 \leq t \leq q, 1 \leq u \leq q^2-2$	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^2-1)(q+1)}$
$u \not\equiv 0 \pmod{q-1}$	
$\chi_{(q+1)(q^2-1)}^{(t)}, t \not\equiv 0 \pmod{q^2-q+1};$	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^3+1)}$
if $t_1 \equiv tq^2$ or $t_2 \equiv tq^4 \pmod{q^3+1}$, then $\chi^{(t)} = \chi^{(t_1)} = \chi^{(t_2)}$	

It therefore suffices to show that $\chi_{(n-3,2,1)}$ is the only irreducible character with degree in the range

$$D_\alpha(1) - d \leq \chi(1) \leq D_\alpha(1).$$

We first consider the slightly easier case of unipotent characters, where we will need the following result.

Lemma 4.20. ([55], 2.1). Let $2 \leq a_1 < a_2 < \dots < a_l$ be integers, $\epsilon_1, \dots, \epsilon_l \in \{1, -1\}$. Then

$$\frac{1}{2} < \frac{(q^{a_1} + \epsilon_1) \dots (q^{a_l} + \epsilon_l)}{q^{a_1 + \dots + a_l}} < 2.$$

Lemma 4.21. Let $n \geq 7$. Let $\chi_\mu \in \text{Irr}(G)$ and $|\chi_\mu(1) - \chi_{(n-3,2,1)}(1)| \leq d$. Then $\mu = (n-3, 2, 1)$.

Proof. For ease we adopt the notation of [55] in this proof, reversing the order in which partitions are written.

Let χ_μ be the unipotent character corresponding to the partition $(\mu_1, \dots, \mu_m) \vdash n$ and set $\lambda_i = \mu_i + i - 1$ for all i . The degree formula (3) in Section 4 can be rewritten ([55], 4.2A) in terms of λ_i to give

$$\chi_\mu(1) = \frac{(q+1)(q^2-1) \dots (q^n - (-1)^n) \prod_{i' < i} (q^{\lambda_i} - (-1)^{\lambda_i + \lambda_{i'}} q^{\lambda_{i'}})}{\left(q^{\binom{m-1}{2} + \binom{m-2}{2} + \dots}\right) \prod_i \prod_{k=1}^{\lambda_i} (q^k - (-1)^k)}. \quad (12)$$

To prove the Lemma we shall use this form of the degree to show that if $\mu \neq (1, 2, n-3)$ then $|\chi_\mu(1) - \chi_{(1,2,n-3)}(1)| > d$. Firstly, the cases $n = 7, 9$ can be checked directly so we may assume that $n \geq 11$. Furthermore we can check explicitly that the statement holds when $\mu_m \geq n-3$ and hence we assume that $\mu_m \leq n-4$. We show that $\chi_\mu(1) \geq q^{4n-17}$, using induction on the length of the partition, m . The conclusion will then follow as $q^{4n-17} - \chi_{(1,2,n-3)}(1) > d$ for $n \geq 9$.

The base case for the induction is the set of two-part partitions, namely $\mu = (k, n-k)$ where $k \geq 4$. When $k = 4$,

$$\chi_{(4,n-4)}(1) = \frac{q^4(q^n+1)(q^{n-1}-1)(q^{n-2}+1)(q^{n-7}-1)}{(q^4-1)(q^3+1)(q^2-1)(q+1)}$$

and it is an easy check that this is greater than q^{4n-17} . When $k \geq 5$, $\chi_\mu(1)$ is

$$\begin{aligned} & \frac{(q+1)(q^2-1) \dots (q^n+1)(q^{n-k+1}+q^k)}{(q+1)(q^2-1) \dots (q-(-1)^k) \cdot (q+1)(q^2-1) \dots (q^{n-k+1}-(-1)^{n-k+1})} \\ &= \frac{(q^{n-k+2}-(-1)^{n-k+2}) \dots (q^n+1)}{(q^2-1) \dots (q-(-1)^k)} \cdot \frac{q^k(q^{n-2k+1}-1)}{q+1} \\ &> q^k \frac{(q^{n-k+2} \cdot q^{n-k+3} \cdot \dots \cdot q^n)/2}{2q^2 \cdot q^3 \cdot \dots \cdot q^k} \geq q^{(k-2)(n-k)+n-2}. \end{aligned}$$

Here we have used Lemma 4.20. As $k \geq 5$ it follows that

$$(k-2)(n-k) + (n-2) - (4n-17) = (n-k)(k-5) + 3(k+5) > 0$$

which gives the conclusion for two-part partitions.

For the induction step assume that μ has length $m \geq 3$. Denote $\mu_1 = k$ and note that the conditions $m \geq 3$ and $n \geq 11$ imply $k \leq n-8$. If μ is $(2^2, n-4)$, $(1, 3, n-4)$, $(2, 3, n-5)$, $(3^2, n-6)$, $(1^2, 2, n-4)$ or $(1^4, n-4)$ we can check directly that $\chi_\mu(1) > q^{4n-17}$. Hence we assume that μ is not of this form, in addition to the original assumption that $\mu_m \leq n-4$. Consider the partition $\nu \vdash n-k$ of length $m-1$ defined by $\nu_i = \mu_{i+1}$. By the above exclusions, it follows that $\nu_{m-1} \leq n-k-4$ and therefore $\chi_\nu(1) > q^{4(n-k)-17}$ by the induction hypothesis. From (12) it also follows that

$$\frac{\chi_\mu(1)}{\chi_\nu(1)} = \frac{\prod_{i=n-k+1}^n (q^i - (-1)^i) \cdot \prod_{i=2}^m (q^{\lambda_i} - (-1)^{\lambda_i+k} q^k)}{\prod_{i=1}^n (q^i - (-1)^i) \cdot \prod_{i=2}^m (q^{\lambda_i} - (-1)^{\lambda_i})}.$$

Note that

$$\lambda_m = \mu_m + m - 1 \leq \mu_m + \mu_1 + \mu_2 + \dots + \mu_{m-1} = n$$

and hence

$$2 \leq \lambda_3 - k \leq \dots \leq \lambda_m - k \leq n - k.$$

We can therefore apply Lemma 4.20 to the collections $(a_1, \dots, a_l) = (\lambda_3 - k, \dots, \lambda_m - k, n - k + 1, \dots, n)$ and $(a_1, \dots, a_l) = (2, 3, \dots, k, \lambda_2, \dots, \lambda_m)$. Also using the inequality

$$(q^{\lambda_2} - (-1)^{\lambda_2+k} q^k)/(q+1) \geq q^{\lambda_2-1}/2,$$

we have

$$\frac{\chi_\mu(1)}{\chi_\nu(1)} = \frac{\frac{1}{2}q^{\lambda_2-1} \cdot \frac{1}{2}q^{\lambda_3} \dots q^{\lambda_m} \cdot q^{n-k+1} \dots q^n}{2q^2 \dots q^k \cdot q^{\lambda_2} \dots q^{\lambda_n}} = \frac{q^{k(n-k)}}{8} \geq q^{k(n-k)-3}.$$

Thus $\chi_{1,\mu}(1) > q^{(4+k)(n-k)-20} = q^{(4n-17)+(k(n-k)-4k-3)} > q^{4n-17}$ as $n - k \geq 8$ and the induction is complete. \square

We next move on to proving the analogous result for non-unipotent characters.

Lemma 4.22. *Let $n \geq 7$ be odd. Let $\chi \in \text{Irr}(SU_n(q))$ be a non-unipotent character such that $|\chi(1) - \chi_{(n-3,2,1)}(1)| \leq d$. Then $(n, q) = (7, 2)$ and*

$$\chi(1) \in \{6622, 10\,234, 9030\}.$$

Proof. Let $\chi = \chi_{s,\bar{\lambda}}$ where $1 \neq s \in G^*$ as defined in Section 4.1. Following ([55], 4.1), let $\tilde{G} = GU_n(q)$ and consider an inverse image $\tilde{s} \in \tilde{G}$ of s . Let C be the complete inverse image of $C^* = C_{G^*}(s)$ in \tilde{G} and set $D = C_{\tilde{G}}(\tilde{s})$. Note that $(G^* : C^*) = (\tilde{G} : C)$ and D is a normal subgroup of C such that $C/D \hookrightarrow C_{q+1}$. Thus by Proposition 3.2,

$$\chi(1) \geq (G^* : C^*)_{p'} \geq (\tilde{G} : D)_{p'}/(q+1).$$

Now \tilde{G} acts on the natural module $V = \mathbb{F}_{q^2}^n$ and we denote the characteristic polynomial of \tilde{s} in this action by $P(t) \in \mathbb{F}_{q^2}[t]$. Then $P(t)$ is a product $\prod_{i=1}^l f_i(t)^{m_i}$ of distinct irreducible polynomials $f_i(t)$ over \mathbb{F}_{q^2} , $1 \leq i \leq l$.

Firstly consider the case where $l = 1$. It follows from the non-triviality of s that $f = f_1$ has degree $k \geq 2$. If $\lambda \in \overline{\mathbb{F}}_q$ is a root of $f(t)$, then $\mathbb{F}_{q^2}(\lambda) = \mathbb{F}_{q^{2k}}$ and the roots $\lambda_j = \lambda^{q^{2(j-1)}}$ of $f(t)$ are distinct for $1 \leq j \leq k$. From ([55], 4.2B) we find further that k is odd and that $D = GU_m(q^k)$ such that $mk = n$.

Thus by Lemma 4.20

$$\begin{aligned} \chi(1) &\geq \frac{(q+1)(q^2-1) \dots (q^{mk} - (-1)^{mk})}{(q+1)(q^k+1) \dots (q^{mk} - (-1)^m)} \\ &> \frac{\frac{1}{2}q^{2+3+\dots+mk}}{2q^{k+2k+\dots+mk}} = \frac{q^{\frac{1}{2}m^2k(k-1)-1}}{4}. \end{aligned}$$

Note that $\chi_{(n-3,2,1)}(1) + d < q^{3n-7}$ for $n \geq 7$. Thus it suffices to show that $q^{\frac{1}{2}m^2k(k-1)-3} \geq q^{3n-7}$. As $k \geq 3$, $\frac{m^2k(k-1)}{2} = n \cdot \frac{n-m}{2} \geq \frac{n^2}{3}$ and then $\frac{n^2}{3} \geq 3n-4$ when $n \geq 9$. When $n = 7$, it follows that $k = 7$ and we still have $\frac{m^2k(k-1)}{2} - 3 \geq 3n-7$. This concludes the case where $l = 1$ and hence we assume that $l > 1$ from now on.

Here we see that D preserves a non-trivial decomposition $V = \bigoplus_{i=1}^l V_i$ where $V_i = P_i(s)(V)$ and $P_i(t) = P(t)/f_i(t)^{m_i}$. We denote $n_i = \dim V_i = m_i \deg(f_i)$ and without loss of generality we assume that $\dim V_1 \leq \dots \leq \dim V_l$.

Suppose D fixes a non-zero totally isotropic subspace W in V . Denote $\dim W = k$ and $b_k = (\tilde{G} : \tilde{G}_W)_{p'}$. Clearly $1 \leq k \leq \frac{n-1}{2}$, and we check using Lemma 4.20 that if $k \geq 2$ then $b_k \geq \min\{b_2, b_{(n-1)/2}\}$. Hence if $k \geq 2$, then we have

$$\chi(1) \geq \min\{b_2, b_{(n-1)/2}\} > \chi_{(n-3,2,1)}(1) + d.$$

We may therefore assume that any D -invariant totally isotropic subspace has dimension 1.

Similarly, if D preserves a degenerate subspace $W \subset V$, we may assume that W is totally isotropic and of dimension 1: firstly note that if $U = W \cap W^\perp$, then $\dim U = 1$ by the above. Let e denote an isotropic basis vector for U and let $v_j \in V$ such that e, v_1, \dots, v_k is a basis for W , where $\langle v_1, \dots, v_k \rangle$ is non-degenerate. Note that $D \leq C_{\tilde{G}}(W) \leq C_{\tilde{G}}(\langle e \rangle) = P_1$, where P_1 is maximal parabolic. As $C_{\tilde{G}}(W)$ stabilises the non-degenerate k -space $W/\langle e \rangle$, it follows that $C_{\tilde{G}}(W) \leq C_{P_1}(W/\langle e \rangle)$. Hence

$$\chi(1) > \frac{|GU_n(q)|_{p'}}{|GU_k(q)|_{p'}|GU_{n-2-k}(q)|_{p'}|GL_1(q^2)|_{p'}(q+1)}.$$

But the right hand side of the above is greater than $\chi_{(n-3,2,1)}(1) + d$ when $k \geq 1$. Hence we may assume that $W = U$ is totally isotropic.

Lastly, if D preserves a non-degenerate subspace U of V , where $1 \leq k = \dim U \leq \frac{n}{2}$, then we can assume $k \leq 3$: if D preserves such a subspace then $D \leq GU_k \times GU_{n-k}$ and $\chi(1)$ is at least

$$\frac{\prod_{i=1}^n (q^i - (-1)^i)}{(q+1) \prod_{i=1}^k (q^i - (-1)^i) \prod_{i=1}^{n-k} (q^i - (-1)^i)} = \frac{\prod_{i=n-k+1}^n (q^i - (-1)^i)}{(q+1) \prod_{i=1}^k (q^i - (-1)^i)}.$$

Now if $k \geq 4$,

$$\frac{\prod_{i=n-k+1}^n (q^i - (-1)^i)}{(q+1) \prod_{i=1}^k (q^i - (-1)^i)} \geq q^{3n-7}$$

and so $\chi(1) > \chi_{1,(n-3,2,1)} + d$ as desired.

It follows from the above assumptions that each V_i is totally isotropic of dimension 1 or non-degenerate of dimension 1, 2, 3, $n-3$, $n-2$ or $n-1$. After a number of very similar calculations when $n = 7$ or 9 we may also assume that $n_l \geq n-3$ and that D preserves no non-degenerate subspaces of V_l . Clearly then $l \leq 4$ and $\max\{n_i\}_{i=1}^{l-1} \leq 3$.

There now remain a number of subcases to consider. Firstly consider the case where $l = 2$, $n_1 = 3$ and $n_2 = n-3$, and D preserves no non-degenerate subspaces of V_1 . If $\tilde{s}|_{V_2}$ is not a scalar it follows from the $l = 1$ case that $D \leq GU_3(q) \times GU_m(q^k)$ where $mk = n-3$ and $k \geq 3$. Consequently

$$\begin{aligned} \chi(1) &> \frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q+1)(q^2-1)(q^3+1)} \cdot \frac{(q+1)(q^2-1) \dots (q^{mk}-1)}{(q+1)(q^k+1) \dots (q^{mk}-(-1)^m)} \\ &> \frac{q^{3n-8}}{8(q+1)} \cdot q^{\frac{1}{2}m^2k(k-1)-1} > q^{3n-14+\frac{1}{2}m^2k(k-1)} \end{aligned}$$

by Lemma 4.20. Evidently $\frac{m^2k(k-1)}{2} \geq m^2k = (n-3)k \geq 2(n-3) \geq 7$ and hence $\chi(1) > q^{3n-7}$. Hence we assume that \tilde{s} does act as a scalar on V_2 . As $3 \neq n-3$, we conclude that either $C = D = GU_3(q) \times GU_{n-3}(q)$ if $\tilde{s}|_{V_1}$ is a scalar, or $C = D = GU_1(q^3) \times GU_{n-3}(q)$ otherwise. Hence

$$\chi(1) \geq \psi(1) \cdot \frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)}{(q^3+1)(q^2-1)(q+1)}$$

where $\psi(1)$ is the degree of the unipotent character of C^* corresponding to χ as in Proposition 3.2. Now if $\psi|_{GU_{n-3}(q)}$ is non-trivial, it has degree at least $\frac{q(q^{n-4}+1)}{(q+1)}$ by ([55], 4.1) and $\chi(1) > \chi_{(n-3,2,1)} + d$, as required. To conclude this case, we assume that $\psi|_{GU_{n-3}(q)}$ is trivial, and then it is easy to find all possible degrees $\psi(1)$ as the unipotent character degrees of $GU_1(q^3)$ and $GU_3(q)$ are known ([8], 13.8). With one exception we find that $d > |\chi(1) - \chi_{1,(n-3,2,1)}|$. The exceptional case occurs when $(n, q) = (7, 2)$, $C = GU_3 \times GU_4(q)$ and ψ corresponds to the multi-partition $((1, 2), (4))$. Here $\psi(1) = \frac{2 \cdot (2^2-1)}{(2+1)}$ and $\chi(1) = \frac{2 \cdot (2^7+1)(2^6-1)(2^5+1)}{(2^3+1)(2+1)^2} = 6622$ as in the conclusion of the Lemma.

Next consider the case where $\max\{n_i\}_{i=1}^{l-1} = 2$. By the work above, it follows that $D \leq GU_2(q) \times GU_{n-2}(q)$ or $D \leq GU_1(q) \times GU_2(q) \times GU_{n-3}(q)$. Assuming the former case for now, if $\tilde{s}|_{V_2}$ is non-scalar then $D \leq GU_2(q) \times GU_m(q^k)$ such that $m = \frac{n-2}{k}$ and $k \geq 3$. Here

$$\begin{aligned} \chi(1) &> \frac{(q^n + 1)(q^{n-1} - 1)}{(q^2 - 1)(q + 1)^2} \cdot \frac{(q + 1) \dots (q^{n-2} + 1)}{(q + 1)(q^k + 1) \dots (q^{n-2} - (-1)^{\frac{n-2}{k}})} \\ &> \frac{q^{2n-3}}{8(q + 1)^2} \cdot q^{\frac{1}{2}m^2k(k-1)-1} > q^{3m-7}. \end{aligned}$$

This leaves the case where $\tilde{s}|_{V_2}$ is a scalar. Assuming additionally that $\tilde{s}|_{V_1}$ is also a scalar, then as $n_1 < n_2$ it follows that $C = D = GU_2(q) \times GU_{n-2}(q)$ and

$$\chi(1) = \psi(1) \cdot \frac{(q^n + 1)(q^{n-1} - 1)}{(q^2 - 1)(q + 1)}.$$

Suppose that $\psi|_{GU_{n-2}(q)}$ is non-trivial. Then by ([55], 4.1), $\psi(1) \geq \frac{q(q^{n-3}-1)}{(q+1)}$ and hence

$$\chi(1) \geq \frac{q(q^n + 1)(q^{n-1} - 1)(q^{n-3} - 1)}{(q^2 - 1)(q + 1)}.$$

We can then check that this is greater than $\chi_{(n-3,2,1)}(1) + d$ as required, provided $(n, q) \neq (7, 2)$. If however $(n, q) = (7, 2)$ and $\psi(1) = \frac{2 \cdot (2^7-3-1)}{(2+1)}$ then

$$|\chi(1) - \chi_{(4,2,1)}(1)| = \left| \frac{2 \cdot (2^7 + 1)(2^6 - 1)(2^4 + 1)}{(2^2 - 1)(2 + 1)^2} - \chi_{(4,2,1)}(1) \right| \leq d$$

and we have another of the listed exceptions to the Lemma. Lastly, if $\psi|_{GU_{n-2}(q)} = 1$ then

$$\chi(1) \in \left\{ \frac{q(q^n + 1)(q^{n-1} - 1)}{(q^2 - 1)(q + 1)}, \frac{(q^n + 1)(q^{n-1} - 1)}{(q^2 - 1)(q + 1)} \right\}$$

and $\chi_{(n-3,2,1)}(1) - \chi(1) > d$. The remaining case where $\tilde{s}|_{V_1}$ is non-scalar and $D \leq GL_1(q^2) \times GU_{n-2}(q)$ follows similarly to the above. Indeed, if $\psi|_{GU_{n-2}(q)}$ is non-trivial then

$$\chi(1) \geq \frac{q(q^n + 1)(q^{n-1} - 1)(q^{n-3} - 1)}{(q^2 - 1)} > \chi_{(n-3,2,1)}(1) + d.$$

If however $\psi|_{GU_{n-2}(q)} = 1$ then

$$\chi(1) = \frac{(q^n + 1)(q^{n-1} - 1)}{(q^2 - 1)(q + 1)} < \chi_{(n-3,2,1)}(1) - d.$$

To complete the proof we require a consideration of the cases where $D \leq GU_1(q) \times GU_2(q) \times GU_{n-3}(q)$ or $D \leq GU_1(q)^i \times GU_{n-i}(q)$ for $i = 1, 2$ or 3 . We shall however omit the details as the proof follows very similarly to the work above. In summary, we find that apart for one exception, $|\chi_{(n-3,2,1)}(1) - \chi(1)| > d$. The single exception occurs when $(n, q) = (7, 2)$, $C = D = GU_1(q)^2 \times GU_5(q)$ and ψ corresponds to the multi-partition $((1), (1), (1, 4))$. In this case $\chi(1) = 9030$. \square

Proof of Proposition 4.19:

Proof. Recall that $\alpha = \chi_{q^2-q}^{(q+1)} \in \text{Irr}(GU_3(q))$. Firstly note from (11) in the proof of Proposition 4.18 that

$$|D_\alpha^\circ(1) - \chi_{(n-3,2,1)}(1)| \leq (\omega|_G, \zeta_{n,q}) \zeta_{n,q}(1) / \alpha(1) = \frac{q^n(q^3 + 1)(q + 1)}{q(q - 1)}.$$

It then follows from Lemmas 4.21 and 4.22 that either $D_\alpha = D_\alpha^\circ = \chi_{(n-3,2,1)}$ as required, or $(n, q) = (7, 2)$ and

$$D_\alpha^\circ(1) \in \{6622, 10\,234, 9030\}.$$

Assume the latter statement for a contradiction and first note that here $D_\alpha(1) = \chi_{(4,2,1)}(1) = 7568$. Therefore as $D_\alpha(1) \geq D_\alpha^\circ(1)$, we can discard the second and third possibilities listed above as they are too large and it follows that $D_\alpha^\circ(1) = 6622$.

From Table 1 we see that $SU_7(2)$ has at least two additional characters of degree 6622. Indeed, when $\beta = \chi_{q^2-q}^{(t)}$ where $t = 1, 2$, we have $D_\beta(1) = 6622$. Furthermore the irreducible component $D_\beta^\circ \in \text{Irr}(SU_7(2))$ has degree lying in the range $4894 \leq D_\beta^\circ(1) \leq 6622$ by Proposition 4.18. But the irreducible character degrees of $SU_7(2)$ are known ([40]) and we check that there exist exactly two irreducible characters with degrees in this range, a contradiction. \square

Lemma 4.23. *Let x and y be the regular semisimple elements chosen in Proposition 4.13. Then $\chi_{(n-3,2,1)}(x) = \chi_{(n-3,2,1)}(y) = 1$. Also, $|\chi_{(3,2,1^{n-5})}(x)| = |\chi_{(3,2,1^{n-5})}(y)| = 1$.*

Proof. By Proposition 4.19, $\chi_{(n-3,2,1)} = D_\alpha$ where $\alpha = \chi_{q^2-q}^{(q+1)} \in \text{Irr}(GU_3(q))$. The values of α are known ([15], Table 7), and recall for $g \in G$ we have the formula

$$D_\alpha(g) = \frac{1}{|S|} \sum_{h \in S} \overline{\alpha(h)} \omega(hg),$$

where $\omega = \zeta_{3n,q}$ is the reducible Weil character of $GU_{3n}(q)$. Applying this formula to x and y then gives the values for $\chi_{(n-3,2,1)}$. It follows from the formula above that $\chi_{(n-3,2,1)}$ is \tilde{G} -invariant and hence extends to a character of \tilde{G} by Lemma 4.7. The values of $|\chi_{(3,2,1^{n-5})}(x)|$ and $|\chi_{(3,2,1^{n-5})}(y)|$ then follow by Lemmas 4.5 and 4.6, as in the proof of Proposition 4.10. \square

4.3.2. Calculating Structure Constants for $G = SU_n(q)$, $n \geq 7$ odd. We now have sufficient information about the characters in the set $X \subset \text{Irr}(\tilde{G})$ defined after Proposition 4.14 to calculate structure constants for G . For $g \in G$, let $g = su$ denote the Jordan decomposition of g with s semisimple and u unipotent. We use the shorthand notation $u \sim (n^{r_n}, \dots, 2^{r_2}, 1^{r_1})$ to indicate the Jordan form of u and let σ_g denote the spectrum of eigenvalues of g . The elements $x, y \in G$ will always denote those chosen in Proposition 4.13. In this section we first prove the following result.

Proposition 4.24. *Let $G = SU_n(q)$ where $n \geq 7$ is odd and let $g = su \in G$. Assume that $u \neq 1$ and also that $u \sim (2, 1^{n-2})$ when $s = 1$. Then $g \in x^G y^G$.*

For ease of presentation we treat the cases where g is unipotent and non-unipotent separately.

Lemma 4.25. *Let $1 \neq u \in G$ be unipotent, and assume that $u \sim (2, 1^{n-2})$. Then $u \in x^G y^G$.*

Proof. By Theorem 2.7 and Proposition 4.14, $u \in x^G y^G$ if and only if

$$\kappa(x^G, y^G, u^G) = \sum_{\chi \in X \subset \text{Irr}(G)} \frac{\chi(x)\chi(y)\chi(u^{-1})}{\chi(1)} \neq 0.$$

Removing the contribution of the trivial character, it suffices to show that

$$\sum_{\chi \in X \setminus \{1_G\}} \frac{|\chi(x)\chi(y)\chi(u^{-1})|}{\chi(1)} < 1. \quad (13)$$

The values of $|\chi(x)\chi(y)|$ are known by Lemmas 4.16 and 4.23, and in all cases $|\chi(x)\chi(y)| \leq 1$. Furthermore, when $q = 2$, $\chi(y) = 0$ for all non-unipotent characters $\chi \in X \setminus \{1_G\}$

(Lemma 4.16). Hence for (13) to hold it is sufficient to show that

$$\sum_{\chi \in X \setminus \{1_G\} \subset \text{Irr}(G)} \frac{|\chi(u)|}{\chi(1)} < 1, \text{ if } q \geq 3,$$

and

$$\frac{|\chi_{(n-3,2,1)}(u)|}{\chi_{(n-3,2,1)}(1)} + \frac{|\chi_{(3,2,1^{n-5})}(u)|}{\chi_{(3,2,1^{n-5})}(1)} + \frac{|St(u)|}{St(1)} < 1, \text{ if } q = 2.$$

In either case, denote the sum of character ratios on the left hand side by $\Delta(u)$.

To prove that $\Delta(u) < 1$, we bound the character ratio summands on a case by case basis. Firstly consider the possible values of $\chi_{(n-3,2,1)}(u)$. Let $u \sim (n^{r_n}, \dots, 2^{r_2}, 1^{r_1})$ and denote the total number of blocks by r . Recall from Proposition 4.19 that $\chi_{(n-3,2,1)} = D_\alpha$ where $\alpha = \chi_{q^2-q}^{(q+1)} \in \text{Irr}(GU_3(q))$ and hence

$$D_\alpha(u) = \frac{1}{|GU_3(q)|} \sum_{h \in GU_3(q)} \overline{\alpha(h)} \omega(hu). \quad (14)$$

Here ω denotes the reducible Weil character of $GU_{3n}(q)$ that takes values

$$\omega(hu) = -(-q)^{\dim \text{Ker}(hu-1)}.$$

The values of α are known ([15], Table 7) and we compute

$$\begin{aligned} |GU_3(q)| \cdot \chi_{(n-3,2,1)}(u) &= (q^2 - q) (-(-q)^{3r} - q) \\ &- q (-(-q)^{3r-r_1} - q) (q-1)(q^3 + 1) \\ &- q^2(q-1)(q^2 - q + 1) (-q^{2r+1} - (-q)^{r+1} - q(q-1)) \\ &+ q^2(q-1)(q^3 + 1) (-(-q)^{2r-r_1+1} - (-q)^{r+1} - q(q-1)) \\ &+ 2q^3(q-1)^2(q^2 - q + 1) \left(\frac{-3(-q)^{r+1} - q(q-2)}{6} \right) \\ &+ \frac{q^4}{3}(q+1)^3(q-1)^2. \end{aligned} \quad (15)$$

Recall that $\chi_{(n-3,2,1)}(1) \sim q^{3n-7}$. Hence when r is large, $\frac{|\chi_{(n-3,2,1)}(u)|}{\chi_{(n-3,2,1)}(1)} \sim q^{3r-3n}$ and checking small cases computationally we see that $\frac{|\chi_{(n-3,2,1)}(u)|}{\chi_{(n-3,2,1)}(1)} < q^{-3}$.

We do not have a method for computing general character values of $\chi_{(3,2,1^{n-5})}$ but it is enough to use the trivial bound $|\chi_{(3,2,1^{n-5})}(g)| \leq |C_G(g)|^{\frac{1}{2}}$. Note that $\chi_{(3,2,1^{n-5})}(1) \sim q^{\frac{1}{2}(n^2-n-8)}$ and unipotent centralizers can be easily computed using the details in ([36], 7.1). Hence we can bound the character ratio $\frac{|\chi_{(3,2,1^{n-5})}(u)|}{\chi_{(3,2,1^{n-5})}(1)}$. These bounds will be useful for later computations so we list some details in Table 2. Note that although we have assumed $u \sim (2, 1^{n-2})$ in the statement of the Lemma, bounds on these elements will be useful later so they are included for completeness.

For the unipotent elements not listed in Table 2, we see that $|C_G(u)|^{\frac{1}{2}} \lesssim q^{\frac{1}{2}(n^2-6n+17)}$. Checking small dimensions computationally we find in these cases that $\frac{|\chi_{(3,2,1^{n-5})}(u)|}{\chi_{(3,2,1^{n-5})}(1)} < \frac{1}{q^3}$.

Next we consider the contribution to $\Delta(u)$ of any character

$$\chi \in \{\chi_{st, (1^{n-1})}, \chi_{st, (2, 1^{n-2})} \mid 1 \leq t \leq q\} \subset X.$$

As stated above, these characters all vanish when $q = 2$ but in general we use the centralizer bound once more. In particular note that $\chi(1) > q \cdot \chi_{(3,2,1^{n-5})}(1)$ and hence the bounds found above for $\frac{|\chi_{(3,2,1^{n-5})}(u)|}{\chi_{(3,2,1^{n-5})}(1)}$, also apply to $q \cdot \frac{|\chi(u)|}{\chi(1)}$. It follows that the sums

TABLE 2. Character Ratio Bounds

Block Structure	Conditions	Upper bound for $\frac{ \chi_{(3,2,1^{n-5})}(u) }{\chi_{(3,2,1^{n-5})}(1)}$
$(2, 1^{n-2})$	$n \geq 9, q = 2, 3$	$\frac{1}{q^2}$
	$n \geq 9, q \geq 4$	$\frac{1}{q^3}$
$(2^2, 1^{n-4})$	$n = 7, q = 2$	0.263
	$n = 7, q \geq 3$	$\frac{1}{q^2}$
	$n \geq 9$	$\frac{1}{q^4}$
$(3, 1^{n-3})$	$n = 7$	$\frac{1}{q^2}$
	$n \geq 9$	$\frac{1}{q^3}$

$\sum_{t=1}^q \frac{|\chi_{s^t, (1^{n-1})}(u)|}{\chi_{s^t, (1^{n-1})}(1)}$ and $\sum_{t=1}^q \frac{|\chi_{s^t, (2, 1^{n-2})}(u)|}{\chi_{s^t, (2, 1^{n-2})}(1)}$ are bounded by the values given in Table 2, or q^{-3} otherwise.

The remaining characters to consider are the non-unipotent families $\{\chi_{s^t, (n-1)}\}_{t=1}^q$ and $\{\chi_{s^t, (n-2, 1)}\}_{t=1}^q$. Let $\chi \in \{\chi_{s^t, (n-1)}\}_{t=1}^q$. We can calculate character values using Lemma 4.15. We compute some of the largest values and print them below for elements of interest.

$$u \sim (2, 1^{n-2}) : \sum_{t=1}^q \frac{|\chi_{s^t, (n-1)}(u)|}{\chi_{s^t, (n-1)}(1)} = \frac{q(q^{n-1} - 1)}{q^n + 1},$$

$$u \sim (2^2, 1^{n-4}), (3, 1^{n-3}) : \sum_{t=1}^q \frac{|\chi_{s^t, (n-1)}(u)|}{\chi_{s^t, (n-1)}(1)} = \frac{q(q^{n-2} + 1)}{q^n + 1}.$$

For all other unipotent elements it is an easy check that $\sum_{t=1}^q \frac{|\chi_{s^t, (n-1)}(u)|}{\chi_{s^t, (n-1)}(1)} < \frac{1}{q^2}$.

Lastly let $\chi \in \{\chi_{s^t, (n-2, 1)}\}_{t=1}^q$. Again we have an explicit formula for character values, namely formula (8) in Section 4.3.1. We find for $u \sim (n^{r_n}, \dots, 2^{r_2}, 1^{r_1})$ with a total number of blocks r , that

$$\begin{aligned} |GU_2(q)| \cdot \chi_{s^t, (n-2, 1)}(u) &= (q-1)(q^{2r} - 1) \\ &\quad - (q^2 - 1)((-q)^{2r-r_1} - 1) \\ &\quad + q(q-1)((-q)^r(-q+1) + (q-1)). \end{aligned}$$

This is maximal when $u \sim (2, 1^{n-2})$ and in this case

$$\sum_{t=1}^q \frac{|\chi_{s^t, (n-2, 1)}(u)|}{\chi_{s^t, (n-2, 1)}(1)} < q \cdot \frac{1}{q^{15/8}} = \frac{1}{q^{7/8}}.$$

For all other unipotent elements,

$$\sum_{t=1}^q \frac{|\chi_{s^t, (n-2, 1)}(u)|}{\chi_{s^t, (n-2, 1)}(1)} < \frac{1}{q^{5/2}}.$$

Collating these bounds for character ratios, we can compute upper bounds for $\Delta(u)$. Firstly let $q = 2$. Here

$$\Delta(u) = \frac{|\chi_{(n-3,2,1)}(u)|}{\chi_{(n-3,2,1)}(1)} + \frac{|\chi_{(3,2,1^{n-5})}(u)|}{\chi_{(3,2,1^{n-5})}(1)} + \frac{|St(u)|}{St(1)}.$$

Using the bounds above and noting that $St(u) = 0$ ([8], 6.4.7) gives

$$\Delta(u) < 0.263 + \frac{1}{2^3} < 1.$$

Similarly, when $q \geq 3$ the bounds above give

$$\Delta(u) = \sum_{\chi \in X \setminus \{1_G\} \subset \text{Irr}(G)} \frac{|\chi(u)|}{\chi(1)} < \frac{q(q^{n-2} + 1)}{(q^n + 1)} + \frac{3}{q^2} + \frac{1}{q^{5/2}} + \frac{1}{q^3} < 1.$$

□

Lemma 4.26. *Let $g = su \in G$ such that $s, u \neq 1$. Then $g \in x^G y^G$.*

Proof. As explained in the proof of Lemma 4.25, for the result to hold it suffices to show that $\Delta(g) < 1$ where

$$\Delta(g) = \sum_{\chi \in X \setminus \{1_G\}} \frac{|\chi(g)|}{\chi(1)}, \text{ if } q \geq 3,$$

and

$$\Delta(g) = \frac{|\chi_{(n-3,2,1)}(g)|}{\chi_{(n-3,2,1)}(1)} + \frac{|\chi_{(3,2,1^{n-5})}(g)|}{\chi_{(3,2,1^{n-5})}(1)} + \frac{|St(g)|}{St(1)}, \text{ if } q = 2.$$

Firstly consider $\chi \in \{\chi_{(3,2,1^{n-5})}\} \cup \{\chi_{s^t, (1^{n-1})}, \chi_{s^t, (2, 1^{n-2})} \mid 1 \leq t \leq q\} \subset X$. As $|C_G(g)| \leq |C_G(u)|$ we can use the bounds for $\frac{|\chi_{(3,2,1^{n-5})}(g)|}{\chi(1)}$ calculated in Lemma 4.25. Note however that Table 2 does not include a treatment of the case $n = 7$, $u \sim (2, 1^5)$. However when $s \neq 1$ and $u \sim (2, 1^5)$ we can sufficiently bound the character ratio by estimating $|C_G(s)|^{1/2}$ and find that $\frac{|\chi_{(3,2,1^{n-5})}(g)|}{\chi(1)} < \frac{1}{q^2}$ in this case.

The value of $|\chi_{(n-3,2,1)}(g)|$ has a slightly more complicated formula when $s \neq 1$ but we can again find a sufficient bound. Let $\sigma_g = \{\lambda_i\}$ denote the eigenvalues of g and note these are not necessarily in \mathbb{F}_{q^2} . Let $\delta, \rho, \tau \in \mathbb{F}_{q^6}$ be elements of orders $q+1$, q^2-1 and q^3+1 respectively. Then define $\sigma_{g,\delta} := \sigma_g \cap \langle \delta \rangle$, $\sigma_{g,\rho} := \{\sigma_g \cap \langle \rho \rangle\} \setminus \sigma_{g,\delta}$ and $\sigma_{g,\tau} := \{\sigma_g \cap \langle \tau \rangle\} \setminus \sigma_{g,\rho}$. From the presence of the Weil character ω in formula (14), we see that the magnitude of $|\chi_{(n-3,2,1)}(g)|$ is controlled by the dimension of the eigenspaces of each $\lambda_i \in \sigma_{g,\delta}, \sigma_{g,\rho}, \sigma_{g,\tau}$. Let $m_\delta := \max_{\lambda \in \sigma_{g,\delta}} \{\dim(\text{Ker}(g - \lambda I))\}$ and define m_ρ and m_τ similarly. Then

$$|\chi_{(n-3,2,1)}(g)| \sim q^{\max_i \{3m_\delta + 2, 2m_\rho + 5, 3m_\tau + 6\} - 7}.$$

In general this is maximal when g has Jordan normal form $(\lambda J_1^{n-2}) \oplus (\mu J_2)$ for $\lambda, \mu \in \sigma_{g,\delta}$ such that $\lambda \neq \mu$. Checking small values of (n, q) explicitly we find that $\frac{|\chi_{(n-3,2,1)}(g)|}{\chi_{(n-3,2,1)}(1)} \leq \frac{1}{q^3}$.

Finally consider $\chi \in \{\chi_{s^t, (n-1)}\}_{t=1}^q \cup \{\chi_{s^t, (n-2, 1)}\}_{t=1}^q$. Here the character values are given by Lemma 4.15 and formula (8) in Section 4.3.1. Much like the above, $|\chi(g)|$ has a non-trivial contribution for each $\lambda_i \in \sigma_{g,\delta}, \sigma_{g,\rho}$ and is maximised when g has Jordan normal form $(\lambda J_1^{n-2}) \oplus (\mu J_2)$ for $\lambda, \mu \in \sigma_{g,\delta}$ such that $\lambda \neq \mu$. Checking small cases computationally we find that

$$\sum_{t=1}^q \frac{|\chi_{s^t, (n-1)}(g)|}{\chi_{s^t, (n-1)}(1)} = \frac{q^{n-2} + q + 2}{q^n + 1} < q^{-15/8}$$

and

$$\sum_{t=1}^q \frac{|\chi_{s^t, (n-2, 1)}(u)|}{\chi_{s^t, (n-2, 1)}(1)} < \frac{1}{q^2}.$$

We collate the information above to bound $\Delta(g)$. Firstly let $q = 2$. Again, as g is not semisimple, $St(g) = 0$ and it follows that

$$\Delta(g) < 0.263 + \frac{1}{2^3} < 1.$$

Now let $q \geq 3$. By the above it follows that

$$\Delta(g) < \frac{1}{q^{15/8}} + \frac{3}{q^2} + \frac{1}{q^2} + \frac{1}{q^3} < 1.$$

□

Lemmas 4.25 and 4.26 together complete the proof of Proposition 4.24.

We can now finally prove Theorem 4.1 where n is odd, to conclude our work on simple unitary groups.

Proof of Theorem 4.1 (n odd):

Proof. We have already seen that the result holds when $n = 3, 5$ (Lemmas 4.11 and 4.12) so we can assume that $n \geq 7$.

Let $g \in SU_n(q)$ with Jordan decomposition $g = su$. If $u = 1$ then it follows from Theorem 3.5 that $\bar{g} = \bar{s} \in \bar{x}^{\bar{G}}\bar{y}^{\bar{G}}$. Similarly, provided $u \sim (2, 1^{n-2})$ when $s = 1$, $\bar{g} \in \bar{x}^{\bar{G}}\bar{y}^{\bar{G}}$ by Lemma 4.24. In either case it then follows that $\text{iw}(\bar{g}) \leq 4$ by Proposition 4.13. This leaves the unipotent element $g = u \sim (2, 1^{n-2})$. Here we can embed u in a subgroup $SU_3(q)$ when $q > 2$, and $SU_4(q)$ when $q = 2$. It follows that $\text{iw}(u) \leq 4$ by Lemma 4.11 and our earlier work in Section 4.2 on even-dimensional unitary groups. This completes the proof of Theorem 4.1. □

5. EXCEPTIONAL GROUPS OF LIE TYPE

In this final section we consider the involution width of the exceptional groups of Lie type. This will complete our case by case study via the classification of finite simple groups and finish the proof of Theorem 1.

$G = E_8(q)$. The first case $E_8(q)$ is illustrative of the method used for the majority of the exceptional groups. Firstly choose regular semisimple elements $x, y \in G$ of orders $r = \text{ppd}(q, 24)$ and $s = \text{ppd}(q, 20)$ respectively. Then $G \setminus \{1\} \subseteq x^G y^G$ by Theorem 7.6 of [25].

Classes of maximal tori in G correspond to classes in the associated Weyl group $W = W(E_8)$ and orders of such tori are given by Carter [6]. By considering possible orders, we see that x lies in a unique maximal torus T_w such that $|T_w| = \Phi_{24}(q) = q^8 - q^4 + 1$. In this instance T_w corresponds to an element $w \in W$ of order 24.

We claim that x is strongly real. For this it suffices to show there exists an involution in G inverting all elements of T_w . Now $N_G(T_w)/T_w \cong C_W(w)$ and thus contains the coset corresponding to the longest element of the Weyl group, namely $w_0 = -1$. This central element acts by inversion on the torus T_w and hence so does any preimage $n_0 \in N_G(T_w)$. As $|T_w|$ is odd and w_0 is an involution, there exists a preimage n_0 that is also an involution.

The argument for the element y is almost identical. Here y is contained in a torus T_w of order $\Phi_{20}(q) = q^8 - q^6 + q^4 - q^2 + 1$ which is also odd. Hence as before y is strongly real, and so $\text{iw}(G) \leq 4$ as required.

$G = {}^2B_2(2^{2n+1})$, $n \geq 1$. Next consider the Suzuki groups ${}^2B_2(2^{2n+1})$. We take elements x and y both of order $r = \text{ppd}(2^{2n+1}, 4)$ and it follows by Theorem 7.1 of [25] that $G \setminus \{1\} \subseteq x^G y^G$. Suzuki [52] showed that G has $q+3$ conjugacy classes of which only two, containing order 4 elements, are not strongly real. Thus x and y are strongly real, and so $\text{iw}(G) \leq 4$.

$G = G_2(q)$, $q \geq 3$. For $q \neq 4$ we take elements x and y of order $r = \text{ppd}(q, 3)$ and it follows by Theorem 7.3 of [25] that $G \setminus \{1\} \subseteq x^G y^G$. As x and y are regular semisimple they lie in a unique maximal torus T , which by choice of r has size $q^2 + q + 1$. Now $W(G_2) \cong D_{12}$ has longest element $w_0 = -1$ and thus as T has odd order, we follow the argument given in the case of $E_8(q)$ to see that x and y are strongly real. Finally, when $q = 4$ we check using GAP [19] that $G_2(4)$ has involution width 3.

$G = {}^2G_2(q)$. Consider $G = {}^2G_2(q)$ with $q = 3^{2n+1} > 3$. Here we take x of order $r = \text{ppd}(q, 6)$ and it follows by ([25], Thm 7.1) that $G \setminus \{1\} \subseteq x^G x^G$. The full character table is known due to Ward [56] and checking orders, x belongs to one of two classes, namely classes V or W (see [56] for notation). These classes contain elements of orders $q + \sqrt{3q} + 1$ and $q - \sqrt{3q} + 1$ respectively. It is now straightforward to show that x is strongly real by computing structure constants. Letting a be a representative of the single conjugacy class of involutions (this has size $q(q-1)(q+1)$), there exist exactly four characters $\chi \in \text{Irr}(G)$ such that $\chi(a)\chi(x) \neq 0$. By Theorem 2.7

$$\begin{aligned} \kappa(a^G, a^G, x^G) &= \sum_{\chi} \frac{\chi(a)^2 \overline{\chi(x)}}{\chi(1)} \\ &= 1 - \frac{1}{3^{2n+1}} + \frac{3^{2n+1} - 1}{3^n(3^{2n+1} + 1 \pm 3^{n+1})} > 0. \end{aligned}$$

Hence $x \in (a^G)^2$ and it follows that $\text{iw}(G) \leq 4$.

$G = {}^3D_4(q)$. This group is strongly real (see Theorem 2.1).

$G = {}^2F_4(2^{2k+1})'$, $k \geq 0$. Let $G = {}^2F_4(q)'$ with $q = 2^{2k+1}$. If $q = 2$ then $\text{iw}(G) = 3$ by GAP [19], so assume $q > 2$. Lemma 2.13 of [26] shows that G contains regular semisimple elements x of order $r = \text{ppd}(2^{2k+1}, 12)$ and y of order $s = \text{ppd}(2^{2k+1}, 6)$, such that $x^G y^G = G \setminus \{1\}$. In the notation of Shinoda [47], x is conjugate to t_{16} or t_{17} and y is conjugate to t_{15} . These lie in unique maximal tori, namely $T(10) \cong \mathbb{Z}_{q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1}$ or $T(11) \cong \mathbb{Z}_{q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1}$ and $T(9) \cong \mathbb{Z}_{q^4 - q^2 + 1}$ respectively.

Note that each of these tori is of odd order. Therefore as the Weyl group $W(G) \cong D_{16}$ again contains the central involution $w_0 = -1$ we can follow the argument for $E_8(q)$ to show that x and y are strongly real.

$G = F_4(q)$. Take regular semisimple elements x and y of orders $r = \text{ppd}(q, 12)$ and $s = \text{ppd}(q, 8)$ respectively and it follows that $G \setminus \{1\} \subseteq x^G y^G$ by ([25], 7.6). Note that these are elements of maximal tori of order $|T_1| = \Phi_{12}(q)$ and $|T_2| = \Phi_8(q)$.

If q is even then both tori have odd order and therefore a preimage $n_0 \in G$ of $w_0 = -1$ can also be taken to be an involution. If q is odd then ([49], Thm.2.3.3) shows that all semisimple element of $F_4(q)$ are strongly real and hence the result follows.

$G = E_7(q)$. Let $G = E_7(q)$ and let $r = (\Phi_2(q)\Phi_{18}(q))_{\{2,3\}}$ and $s = \Phi_7(q)$. By ([25], 7.7) there exists regular semisimple x and $y \in G$ of orders r and s respectively, such that $G \setminus \{1\} \subseteq x^G \cdot y^G$. Let $G_{ad} = E_7(q).(2, q-1)$ denote the adjoint group of type E_7 , that is, the simple group extended by an additional diagonal automorphism of order $(2, q-1)$. By ([49], Thm 2.3.3), every semisimple element of G_{ad} is strongly real in G_{ad} . Therefore, when q is even and so $G = G_{ad}$, it follows that x and y are strongly real and $\text{iw}(G) \leq 4$. Now assume that the characteristic is odd. The structures of maximal tori of G_{ad} are given in [11] and we check that $x \in T_{1,ad}$ where $T_{1,ad}$ is cyclic of order $m = (q+1)(q^6 - q^3 + 1)$. Again by ([49], Thm 2.3.3), there exists an involution $n_0 \in G_{ad}$ such that $x^{n_0} = x^{-1}$. In particular, if $\langle t \rangle = T_{1,ad}$ then x is contained in the dihedral subgroup $D_{2m} = \langle t, n_0 \rangle$. Note that $t^k n_0$ is an involution for all k and inverts x by conjugation. Thus if $D_{2m} \cap E_7(q)$ contains such an element then x is strongly real in the simple group. Assuming otherwise, it follows that $D_{2m} \cap E_7(q) \subset T_{1,ad} \cap E_7(q)$ and hence $|D_{2m} \cap E_7(q)| \leq \frac{m}{2}$. This is a contradiction and hence x is indeed strongly real in $E_7(q)$. The same conclusion for y follows identically as the maximal torus $T_{2,ad}$, containing y and of order $q^7 - 1$, is again cyclic.

$G = E_6^\epsilon(q)$. The final families of exceptional groups $E_6(q)$ and ${}^2E_6(q)$ require a more careful consideration. This is because there no longer exists a central element -1 in the Weyl group $W(E_6)$ and we thus have to look harder for strongly real elements. Instead, for root systems of type E_6 (as well as A_l , and D_l for l odd) the longest element w_0 corresponds to the product of -1 and the nontrivial symmetry of the Dynkin diagram.

Let $\mathbf{G} = E_6(\overline{\mathbb{F}}_q)_{ad}$ denote the adjoint algebraic group of type E_6 and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Frobenius endomorphism such that $G = \mathbf{G}^F$ is the finite adjoint group $E_6(q)_{ad}$ or ${}^2E_6(q)_{ad}$. These finite groups are not necessarily simple but the derived groups $G' = E_6^\epsilon(q)$ or ${}^2E_6^\epsilon(q)$ are simple. We will use the notation $E_6^\epsilon(q)$ where $\epsilon \in \{+, -\}$, to denote $E_6(q)$ if $\epsilon = +$ and ${}^2E_6(q)$ if $\epsilon = -$. Define $d := (3, q - \epsilon!)$ and note that $|G : G'| = d$.

In the remainder of this section we prove the following result.

Theorem 5.1. *Every element in $G' = E_6^\epsilon(q)$ can be written as a product of at most 4 involutions.*

As before, the strategy is to pick two strongly real classes $x^{G'}$ and $y^{G'}$ such that the product $x^{G'} \cdot y^{G'}$ covers as much of the group $G' = E_6^\epsilon(q)$ as possible. If however there exists $g \in G'$ for which we cannot show that $g \in x^{G'} \cdot y^{G'}$, then we embed g in a subgroup $X \subset G'$, where X is a Lie type group for which the involution width is already known. Throughout this section we assume that $q \neq 2$. The Theorem can be checked for $E_6^\epsilon(2)$ using GAP [19].

Choose $x, y \in G'$ with x of order $r = \text{ppd}(q, 12)$ and y of order $s = \text{ppd}(q, 8)$. Note that r and s divide $\Phi_{12}(q) = q^4 - q^2 + 1$ and $\Phi_8(q) = q^4 + 1$ respectively. The element x is contained in a Coxeter torus of a subgroup $F_4(q)$, and $|C_{G'}(x)| = (q^4 - q^2 + 1)(q^2 + q + 1)/d$, while y is a regular semisimple element contained in the unique maximal torus $C_{G'}(y) = T$ where $|T| = (q^4 + 1)(q^2 - 1)/d$.

Lemma 5.2. *The elements x and y are strongly real in $G' = E_6^\epsilon(q)$.*

Proof. Firstly x is contained in a maximal torus of the subgroup $F_4(q)$. This torus has odd order $\Phi_{12}(q)$ and hence x is inverted by an involution contained in $F_4(q)$. Therefore x is also strongly real in the full group $E_6^\epsilon(q)$.

We can also embed y in a subgroup, namely a spin group $D_5^\epsilon(q) \subset E_6^\epsilon(q)$. Note that in $D_5^\epsilon(q)$, y is contained in a maximal torus of order $(q^4 + 1)(q + \epsilon)$. Let \bar{y} denote the image of y in $\Omega_{10}^\epsilon(q)$. It then follows that \bar{y} is strongly real in $\Omega_{10}^\epsilon(q)$ by ([41], 2.5(c) and 2.6 (c)) and we let $\bar{t} \in \Omega_{10}^\epsilon(q)$ be an involution inverting \bar{y} . The involutions of $\Omega_{10}^\epsilon(q)$ that lift to

involutions in $D_5^\epsilon(q)$ are those where the dimension of the negative eigenspace is divisible by 4 ([4], 8.4). This is true for \bar{t} in our case as in $SO_{10}(\overline{\mathbb{F}}_q)$, y is conjugate to an element of the form $\text{diag}(\lambda, \lambda^q, \lambda^{q^2}, \lambda^{q^3}, \lambda^{-1}, \lambda^{-q}, \lambda^{-q^2}, \lambda^{-q^3}, 1, 1)$ where $|\lambda| = s$. Hence \bar{t} lifts to an involution $t \in D_5^\epsilon(q)$ that inverts y by conjugation. \square

As usual, to show that a given conjugacy class $g^{G'}$ is contained in the product $x^{G'}y^{G'}$, we compute the structure constants using Theorem 2.7. As in the proof of Theorem 4.1, the first step is to reduce to the case of unipotent characters.

Firstly consider the untwisted case $G' = E_6(q)$. In the following lemma, St denotes the Steinberg character, and there are a further two characters of interest, namely $D_{4,1}$ and $D_{4,\epsilon}$. These characters arise from a cuspidal unipotent character of the Levi subgroup $D_4(q)$ of G' . Full details are available in ([8], Sec. 13.9).

Lemma 5.3. *Suppose $\chi \in \text{Irr}(G')$ such that $\chi(x)\chi(y) \neq 0$. Then χ is unipotent and $\chi \in \{1, St, D_{4,1}, D_{4,\epsilon}\}$.*

Proof. Let $\tilde{G} = E_6(q)_{sc}$ denote the simply connected finite simple group of type E_6 . Let $\tilde{x}, \tilde{y} \in \tilde{G}$ denote preimages of x, y , and let $\tilde{\chi} \in \text{Irr}(\tilde{G})$ such that $\tilde{\chi}(\tilde{x})\tilde{\chi}(\tilde{y}) \neq 0$. Recall that the irreducible characters are partitioned into Lusztig series as described in Section 3.1. By Lemma 3.1 there exists a semisimple class (t) such that $\tilde{\chi} \in \mathcal{E}(\tilde{G}, (t))$ and we firstly assume that $t = 1$. Here, $\tilde{\chi}$ is by definition a unipotent character. The degrees of such characters match those of the unipotent characters of G' by ([13], 13.20) and these degrees are given in ([8], 13.9). It is easy to check that, excluding the characters $1, St, D_{4,1}$ and $D_{4,\epsilon}$, all unipotent characters of G' are of defect zero for either r or s . It therefore follows from Theorem 3.3 that $\tilde{\chi}$ descends to one of these four characters of G' . Now we consider $t \neq 1$. In this case we use the formula given in Proposition 3.2 to find $\tilde{\chi}(1)$ and in particular we note that for a given prime l , $\tilde{\chi}$ will have l -defect zero if $|C_G(t)|_l = 1$. But the orders of centralizers of semisimple elements are known ([12], Table 1 and [10], Table 4) and we check that for all t , $|C_G(t)|_l = 1$ for either $l = r$ or $l = s$. Hence $\tilde{\chi}$ vanishes on either \tilde{x} or \tilde{y} by Theorem 3.3. The statement of the Lemma then follows. \square

Lemma 5.4. *Suppose that $g \in G'$ such that $|C_{G'}(g)| \leq q^{26}$. Then $g \in x^G \cdot y^G$.*

Proof. By Lemma 5.3 we can so far evaluate the normalised structure constant as follows

$$\kappa(x^{G'}, y^{G'}, g^{G'}) = 1 + \sum_{\chi \in X} \frac{\chi(x)\chi(y)\overline{\chi(g)}}{\chi(1)},$$

where $X = \{D_{4,1}, D_{4,\epsilon}, St\}$.

Furthermore these characters have degrees

$$St(1) = q^{36}, \quad D_{4,1}(1) = \frac{q^3}{2}\Phi_1^4\Phi_3^2\Phi_5\Phi_9, \quad D_{4,\epsilon}(1) = \frac{q^{15}}{2}\Phi_1^4\Phi_3^2\Phi_5\Phi_9,$$

where $\Phi_i = \Phi_i(q)$ is the i -th cyclotomic polynomial.

Recall that $St(x), St(y) \in \{\pm 1\}$ ([8], 6.4.7), and also note the trivial bound $|\chi(g)| \leq |C_{G'}(g)|^{\frac{1}{2}} \leq q^{13}$ for all $\chi \in \text{Irr}(G')$. Thus as

$$|C_{G'}(x)| \cdot |C_{G'}(y)| \leq (q^4 + 1)(q^2 - 1)(q^4 - q^2 + 1)(q^2 + q + 1) \leq q^{13},$$

it follows that

$$\left| \sum_{\chi \in X} \frac{\chi(x)\chi(y)\overline{\chi(g)}}{\chi(1)} \right| \leq \frac{q^{\frac{13}{2}}|D_{4,1}(g)|}{\frac{1}{2}q^3\Phi_1^4\Phi_3^2\Phi_5\Phi_9} + \frac{q^{\frac{13}{2}}|D_{4,\epsilon}(g)|}{\frac{1}{2}q^{15}\Phi_1^4\Phi_3^2\Phi_5\Phi_9} + \frac{|St(g)|}{q^{36}}.$$

The right hand side of the above is then strictly less than 1 as $|D_{4,1}(g)|$, $|D_{4,\epsilon}(g)|$ and $|St(g)|$ are all at most q^{13} by the hypothesis $|C_{G'}(g)| \leq q^{26}$. Hence $\kappa(x^G, y^G, g^G) \neq 0$ and $g^{G'} \in x^{G'} y^{G'}$. \square

We can conclude by Lemma 5.2 that if $|C_{G'}(g)| \leq q^{26}$ then g has involution width at most 4. The result also follows if $g = s$ is semisimple as here $g \in x^{G'} y^{G'}$ follows from Theorem 3.5. To prove the result for the remaining $g \in E_6(q)$, we embed the centralizer (and therefore g) in a group of Lie type for which the involution width is already known.

Recall that \mathbf{G} denotes the adjoint algebraic group $E_6(\bar{\mathbb{F}}_q)_{ad}$ and (as we are currently assuming $G' = E_6(q)$), F is the standard Frobenius map.

Lemma 5.5. *Let $g \in E_6(q)$ such that $|C_{G'}(g)| > q^{26}$. Then g has involution width at most 4.*

Proof. Let $g = su$ be the usual Jordan decomposition. By the above we can assume that $u \neq 1$. Furthermore, as $C_{G'}(su) = C_{G'}(s) \cap C_{G'}(u)$, it follows that $|C_{G'}(s)|, |C_{G'}(u)| > q^{26}$.

Case 1: Unipotent elements: Assume $g = u$. The centralizer sizes of unipotent elements are given by Mizuno ([44], Section 4). For a given centralizer, representatives of the associated class are given in terms of products of root elements of G . In particular the \mathbf{G} -class of $g = u$ has one of the following labels:

$$A_1, A_1^2, A_2, A_1^3, A_2 A_1, A_2 A_1^2, A_2^2.$$

In each case, u is a distinguished unipotent element in a Levi subgroup \mathbf{L} of \mathbf{G} corresponding to the label. Denoting $C = C_{\mathbf{G}}(u)$, the number of G -classes in $u^{\mathbf{G}} \cap G$ is equal to the number of classes in C/C^0 by Lang's Theorem ([36], 2.12). In all cases except when u is of type A_2 , $C = C^0$ and hence $u^{\mathbf{G}} \cap G$ is a single class in G . Furthermore, in such cases \mathbf{L} is F -stable and $u \in (\mathbf{L}^F)'$. All such subgroups lie in a subsystem subgroup $A_2(q)^3$ of G and so u is contained in central product of 3 copies of $SL_3(q)$. The result then follows from Theorem 3.6. Note in the case where u has label A_2^2 , $u^{\mathbf{G}} \cap G'$ splits into $d = (3, q-1)$ classes. However because $N_G(A_2(q)^3)$ contains the diagonal automorphism we may still embed all conjugacy class representatives of $u^{\mathbf{G}} \cap G'$ in a subgroup $A_2(q)^3$. This is not of concern otherwise, as it is only in this case mentioned that the class splits in the simple group. Next assume that \mathbf{L} has label A_2 . Here $C/C^0 = \mathbb{Z}_2$ and thus $u^{\mathbf{G}} \cap G$ splits into 2 G -classes. Let us adopt the usual notation where $\alpha = c_1 \dots c_6$ denotes the root $\sum c_i \alpha_i$ (here α_i refer to the fundamental roots of the E_6 system in the usual ordering, see [44]) and $x_\alpha(t)$ is a corresponding root element. Then by [44], there exist representatives of the two classes of the form

$$x_2 = x_{100000}(1)x_{001000}(1),$$

$$x_{21} = x_{100000}(1)x_{000100}(1)x_{000001}(1)x_{122321}(\zeta), \quad p \neq 2,$$

$$x_{40} = x_{010000}(1)x_{001000}(1)x_{000010}(1)x_{010110}(1)x_{011210}(\eta), \quad p = 2,$$

where ζ is a fixed non square and $x^2 + x + \eta$ is irreducible over \mathbb{F}_q . Now x_2 lies in a Levi subgroup $(\mathbf{L}')^F = A_2(q)$ and the result follows as before. For x_{21} , $\alpha = 122321$ denotes the longest root of the E_6 system and hence the four roots span an A_1^4 subsystem. It follows that x_{21} is contained in $A_1(q)^4 \subset D_4(q)$, a spin group. The unipotent element x_{21} is uniquely determined by its Jordan decomposition which has form (J_3^2, J_1^2) on the natural 8-dimensional D_4 -module. Consequently, x_{21} is contained in a subgroup $SL_3(q)$ and so x_{21} has involution width at most 4 by Theorem 3.6. Finally x_{40} is contained in a subgroup $D_4(q)$. This $D_4(q)$ is the orthogonal group in characteristic 2 and the result follows from Theorem 3.8.

Case 2: Non-unipotent elements: Assuming now that s is non trivial, it follows that $C_G(s)$ is a subsystem subgroup of order at least q^{26} . Inspection of such subgroups (see [10]) shows that $C_G(s)$ has a quasisimple normal subgroup

$$C = D_5(q), D_4^\delta(q), A_4(q) \text{ or } A_5(q).$$

Firstly suppose $C = D_5(q)$. Orders of unipotent centralizers in C are given in ([36], Table 8.6a) and are determined by the Jordan block structure of $u \in C$ on the natural 10-dimensional C -module. As $C_G(s) = C \circ (q-1)$, it follows that $|C_C(u)| > q^{25}$ and possible block structures are

$$q \text{ odd: } u = (J_3, J_1^7), (J_2^2, J_1^6) \text{ or } (J_1^{10})$$

$$q \text{ even: } u = (J_2^2, J_1^6) (2 \text{ classes}) \text{ or } (J_1^{10}).$$

In all cases, u is centralised by a fundamental subgroup $A_1(q)$ of C and hence $u \in C_G(A_1(q)) = A_5(q)$. As $s \in Z(C)$, it follows that $g = su \in A_5(q) = SL_6(q)/Z$ where $Z \subseteq Z(SL_6(q))$ (see [37], Table 5.1) and the result follows from Theorem 3.6. If $C = D_4^\delta(q)$, then we can check possible centralizer dimensions in ([36], 8.6) and the restriction $|C_C(u)| > q^{24}$ forces u to be the identity, a contradiction. The case where $C = A_5(q)$ can be dealt with in a similar manner: unipotent centralizers in linear groups are well known (see for example [36]) and by checking orders it follows that $u = u_1 u_2$ where each u_i is either a transvection or 1. In all cases u is centralised by an $A_1(q)$ subgroup and the result follows as above. The final case where $C = A_4(q)$ follows identically. \square

We now turn to the twisted case and complete the proof of Theorem 5.1 when $G' = {}^2E_6(q)$. Recall that $G^F = G = {}^2E_6(q)_{ad}$ and $G' = {}^2E_6(q)$ with $q > 2$. In the following, $\phi_{8,3'}$ and $\phi_{8,9''}$ denote unipotent characters of G' defined in ([8], Sec. 13.9).

Lemma 5.6. *Suppose $\chi \in \text{Irr}(G')$ such that $\chi(x)\chi(y) \neq 0$. Then χ is unipotent and $\chi \in \{1, St, \phi_{8,3'}, \phi_{8,9''}\}$.*

Proof. Here the proof follows exactly as for Lemma 5.3 by considering $\tilde{G} = {}^2E_6(q)_{sc}$, so we shall be brief. If χ is unipotent then its degree is given in ([8], 13.9) and we check that unless $\chi \in \{1, St, \phi_{8,3'}, \phi_{8,9''}\}$, it is defect zero for r or s . Therefore if χ is not one of these exceptions it will vanish on x or y respectively. All non-unipotent characters will similarly vanish as for all semisimple $1 \neq t \in G$, $|C_G(t)|_l = 1$ for either $l = r$ or $l = s$. \square

Lemma 5.7. *Suppose that $g \in G$ such that $|C_{G'}(g)| \leq q^{28}$. Then $g \in x^G \cdot y^G$.*

Proof. The proof follows exactly as in Lemma 5.4, with an application of Lemma 5.6. We omit the details but note the character degrees

$$St(1) = q^{36}, \phi_{8,3'} = \frac{q^3}{2} \Phi_2^4 \Phi_6^2 \Phi_{10} \Phi_{18}, \phi_{8,9''} = \frac{q^{15}}{2} \Phi_2^4 \Phi_6^2 \Phi_{10} \Phi_{18}.$$

\square

Hence if $|C_{G'}(g)| \leq q^{28}$, then g has involution width at most 4 by Lemma 5.2. Semisimple elements also have involution width at most 4 by Theorem 3.5. The following result handles the remaining cases.

Lemma 5.8. *Let $g \in G'$ such that $|C_{G'}(g)| > q^{28}$. Then g has involution width at most 4.*

Proof. We employ the same method as Lemma 5.5, embedding $g = su$ in a subgroup of ${}^2E_6(q)$ for which an involution width result already exists. As before, we can assume that $u \neq 1$.

Case 1: Unipotent elements: The structure of unipotent centralizers are known in G' ([36] Table 22.1.3). Therefore the restriction $|C_G(u)| > q^{28}$ yields that $g = u$ has one of the following labels in the algebraic group \mathbf{G} :

$$A_1, A_1^2, A_2, A_1^3, A_2A_1, A_2A_1^2, A_2^2.$$

Following the same argument as in the untwisted case, we see that excluding type A_2 , $u^{\mathbf{G}} \cap G$ is a single unipotent class of G . In particular we note that in \mathbf{G} , the elements above are all contained in the subsystem A_2^3 . Thus u is contained in a corresponding finite subsystem subgroup of G' . These are found in ([11], Sec.A(ii)) and the possibilities are $A_2(q)A_2(q^2)$, $A_2^-(q^3)$ or $A_2^-(q)^3$. Hence u is contained in a central product of copies of SL_3 and SU_3 and the result follows by Theorems 3.6 and 4.11. When u has label A_2 , $u^{\mathbf{G}} \cap G$ contains two G -classes. Here the reductive part of $C_{\mathbf{G}}(u)$ is $D = A_2^2$ with $D^F = A_2^-(q)^2$ or $A_2(q^2)$. This has \mathbf{G} -centralizer $\mathbf{L}' = A_2$ so $u \in \mathbf{L}'^F = A_2(q)$ and the result follows from Lemma 4.11.

Case 2: Non-unipotent elements: Assuming now that $g = su$ where s is nontrivial, it follows that $C_G(s)$ is a subsystem subgroup of order at least q^{28} . Such centralizers have been classified and checking against [12] we see that $C_G(s)$ has a quasisimple normal subgroup

$$C = D_5^-(q), D_4^\delta(q), A_4^-(q) \text{ or } A_5^-(q).$$

Now we complete the proof just as in Lemma 5.5. □

This concludes the proof of Theorem 5.1 for the groups $E_6^\epsilon(q)$ and hence the proof of Theorem 1 is complete.

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